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THREE-DIMENSIONAL LUNAR MISSION STUDIES

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Langley Field, Va.

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SUMMARY

Some three-dimensional lunar trajectories have been calculated by integration of the equations of motion of the classical restricted three-body problem of celestial mechanics. The calculations have been used for analysis of several aspects of lunar flight including requirements for achieving lunar impact and for establishment of a close lunar satellite. The allowable errors in initial conditions for lunar missions are strongly dependent on the values of the initial injection velocity and the injection angle. There can be large differences in results obtained from two-dimensional analyses (in which the vehicle trajectory is assumed to remain always in the earth-moon plane) and those obtained from three-dimensional analyses. Some of the accuracy tolerances can be fairly well estimated by use of a two-body analysis which considers the inclination of the plane of the vehicle trajectory to the earth-moon plane. Satisfactory orbits for a relatively close lunar satellite can be obtained with accuracies in the initial conditions approximately equal to those required for lunar impact.

INTRODUCTION

The current literature pertaining to three-body lunar trajectory studies has been almost exclusively confined to planar, or two-dimensional, analyses in which the space vehicle is always contained in the plane of motion of the earth and moon. These two-dimensional studies (for instance, refs. 1 to 4) have been useful in analyzing basic features of the lunar exploration problem; however, they cannot deal with some of the more interesting aspects of the problem. As more sophisticated plans are developed for lunar exploration, the mathematical studies must include more accurate representation of the physical conditions in order to perform realistic trajectory studies. Consideration of the three-dimensionality of the problem should certainly be included at an early date. It is presumed that various organizations have considered three-dimensional trajectory calculations with application to lunar exploration, but very little of this work has been reported in the literature. One exception is reference 5 which contains some comparisons of two- and three-dimensional lunar impact calculations.

Some three-dimensional lunar trajectory studies have been made by the Theoretical Mechanics Division of the Langley Research Center. These studies have been carried out by integration of the equations of motion of the classical restricted three-body problem of celestial mechanics. The integrations were performed by a fourth-order Runge-Kutta numerical integration procedure on an electronic computer, from which was obtained a time history of the space-vehicle velocity and position components. The calculations have been used for analysis of several aspects of lunar flight including requirements for achieving lunar impact and for the establishment of a close lunar satellite.

In any presentation of trajectory calculations it is necessary to choose a fairly limited range of initial conditions for which to present the results. For the lunar exploration problem, estimates of allowable tolerances in the initial conditions can be considerably dependent on the basic values of initial conditions used in the investigation, particularly in the values of injection velocity and injection angle. This limitation has led to fairly sizable differences in some of the tolerance estimates given in the recent literature; these differences are primarily due to differences in values of the injection velocity and injection angle considered. A discussion of the overall effects of injection velocity and injection angle on allowable tolerances is included.

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SYMBOLS

Refer to figures 1 and 2 for illustration of some of the symbols described below.

D	distance from center of earth to center of moon, miles
m_m	mass of moon, $\frac{1}{81.45}$ mass of earth
m_t	total mass of earth and moon
P	sidereal period of moon, based on assumed constant distance from earth to moon, hr
r_1	nondimensional radius from center of earth to space vehicle, $\sqrt{(x - x_1)^2 + y^2 + z^2}$
r_2	nondimensional radius from center of moon to space vehicle, $\sqrt{(x - x_2)^2 + y^2 + z^2}$

t	nondimensional time, time in hours divided by $P/2\pi$
T	time of flight from earth to moon, hr
V	launch velocity (velocity at injection), ft/sec
V_P	parabolic, or escape, velocity at a given radius from earth center (parabolic velocity is 35,384.5 ft/sec at a radius of 4,259 miles, the injection radius for all cases presented in this paper), ft/sec
x_1	nondimensional distance from center of earth to center of mass of earth-moon system, distance in miles divided by D
x_2	nondimensional distance from center of moon to center of mass of earth-moon system, distance in miles divided by D
x,y,z	nondimensional position components of space vehicle measured from center of mass of earth-moon system, distance in miles divided by D
β	heading angle, zero for due east firing, positive for north of east firing, and negative for south of east firing, deg
γ	injection angle, angle between velocity vector and normal to radius vector r_1 at injection point, deg
η	angle between plane of vehicle trajectory and earth-moon plane, deg
θ	firing angle, angle between radius vector r_1 at injection and radius vector to target point, deg
μ	ratio of mass of moon to total mass of earth and moon, m_m/m_t
ψ	position angle, angle between radius vector r_1 at injection and earth-moon axis at injection, measured in earth-moon plane, deg (errors in ψ are related to errors in firing time, a 1° change in ψ being approximately equal to 4 minutes change in injection time)

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TRAJECTORY CALCULATIONS

Equations of Motion

The equations of motion of the restricted three-body problem have been programed on an IBM type 704 electronic data processing machine for step-by-step integration. In these equations, the finite bodies, earth and moon, are considered to rotate in circles about their common center of mass at a uniform angular velocity and the infinitesimal body, the space vehicle, is subject to their gravitational attraction. The xy plane is the plane of motion of the moon around the earth and the axis system is rotating about the origin of the coordinates, such that the x-axis is the line joining the centers of the earth and moon. (See fig. 1.) The distance between the centers of the earth and moon is the unit of distance; the sum of the masses of the earth and moon is the unit of mass; and $P/2\pi$ is the unit of time where P is the sidereal period of the moon. With origin of coordinates at the center of mass of the earth-moon system, the differential equations of motion of the space vehicle are (see ref. 6):

$$\frac{d^2x}{dt^2} - 2 \frac{dy}{dt} = x - (1 - \mu) \frac{(x - x_1)}{r_1^3} - \mu \frac{(x - x_2)}{r_2^3}$$

$$\frac{d^2y}{dt^2} + 2 \frac{dx}{dt} = y - (1 - \mu) \frac{y}{r_1^3} - \mu \frac{y}{r_2^3}$$

$$\frac{d^2z}{dt^2} = -(1 - \mu) \frac{z}{r_1^3} - \mu \frac{z}{r_2^3}$$

where

$$r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}$$

$$\mu = \frac{m_m}{m_t} = \frac{1}{82.45}$$

$$x_1 = -0.0121286 = -\frac{1}{82.45}$$

$$x_2 = 0.9878714 = \frac{81.45}{82.45}$$

With given initial values of velocity and position components, the equations are integrated to give a time history of space-vehicle velocity and position components with respect to the rotating axis system.

Initial Conditions

In the determination of the initial conditions, two-body problem results in the plane of the trajectory were used for preliminary estimates of initial velocity, injection angle, and time of flight. Solutions of spherical triangles utilizing the latitude of the injection point, declination of the moon at contact, and the maximum declination of the moon, in conjunction with a due east heading of the vehicle at injection, provide values of angular relations required for transformation of the preliminary two-body results to the three-dimensional coordinates of the rotating axis system. Small corrections in the initial velocity and/or the heading and position angles as determined from the preliminary two-body results were required for achieving trajectories which would go through the center of the moon or through a desired point with respect to the moon.

Except where otherwise noted, the initial conditions in this paper are calculated for a due east injection from latitude $28^{\circ}27.6'$ North and an injection altitude of 300 statute miles (radius from center of earth of 4,259 statute miles). The distance from the center of the earth to the center of the moon (which is assumed to be constant) is chosen as that for the moon at perigee for the month of September 1959 and is 229,100 statute miles. The maximum declination for that month is $\pm 18.2^{\circ}$. (Data are from ref. 7.) Declination of the moon at time of space-vehicle contact or nearest approach varies between the limits of approximately 12° and 18° South for the examples given in this paper.

The injection angle used in most of the calculations presented in this paper is 25° . This angle was chosen on the basis of consideration of the orientation of the retro-rocket for establishment of a close lunar satellite. With the assumptions that the carrier rocket is spin-stabilized at injection and that the retro-rocket is directed along the spin axis, the retro-rocket orientation will remain fixed in inertial coordinates. Maximum effectiveness of the retro-rocket velocity increment is obtained when the retro-rocket thrust axis is aligned along the velocity

vector at the time of retro-rocket firing. On the basis of some preliminary calculations, an injection angle of 25° appeared to give fair alinement of the retro-rocket axis and the velocity vector at the time of retro-rocket firing. It is not intended to imply that this injection angle is an optimum, and some calculations for an injection angle of 17° are also presented.

Numerical Integration Accuracy

Although it is difficult to ascertain the accuracy of a computation program using step-by-step integration, several tests have been made of the machine program used for the present analysis to indicate whether the accuracy obtained is within that required for the present purposes. The primary sources of inaccuracy in such a program are those due to insufficiently small time intervals and those due to round-off errors for a great number of time intervals. In the present program, the time interval for each step is adjusted in such a way that the difference in results obtained for a step with a certain time interval and for two steps with time intervals of one-half the size of the first is kept within a specified error criterion. As a check on the error criterion, some trajectories were calculated with error criteria one-tenth and one-hundredth of that used for the trajectories presented in this paper, and these trajectories were compared with the original trajectory in the vicinity of the moon. No differences in the trajectories were discernible on a rather large scale (about ten times as large as that in fig. 4). A partial check on round-off error was obtained by running a trajectory backwards in time from a position near the moon and comparing the velocity and position components with the initial values. Reasonably small deviations were noted (of the order of 4 ft/sec in velocity and 1 mile in position). The constant in Jacobi's integral (see ref. 6) was computed and tabulated to help detect any large random error or incorrect input in the program. Finally, it was noted throughout the analysis that small changes in the initial conditions (of the order of 2 or 3 ft/sec for velocity and 1 mile for position) gave consistent variations in the trajectories in the vicinity of the moon. From such investigations, it is concluded that the computation program is sufficiently accurate for the present purposes.

RESULTS AND DISCUSSION

Trajectories and Error Analysis for Lunar Impact

Some typical impact trajectories which are aimed at the center of the moon are shown in figure 3. The trajectories are plotted with respect to the rotating axis system in which the x-axis is the line joining the

centers of the earth and moon and rotates with the moon. For the values of injection angle, injection latitude, and declination of the moon used for these trajectories, it is seen that the space vehicle remains relatively close to the earth-moon plane throughout its travel to the moon.

As in the case of the two-dimensional studies of reference 4, use of the two-body results for initial velocity, injection angle, time of flight, and position angle are generally sufficiently accurate to achieve impact on the moon in the three-dimensional case. (These parameters must, of course, be properly transformed to the coordinate system used.) In order to achieve trajectories aimed at the center of the moon rather than those which simply impact somewhere on the surface, small adjustments in the initial velocities and/or angles obtained from the two-body approximation are sometimes necessary.

An error analysis was conducted to determine the accuracy required in the initial conditions in order to achieve space-vehicle impact on the moon. Individual variations were made in velocity, injection angle, heading angle, and position angle from the basic values for trajectories aimed at the center of the moon. A typical plot of the results obtained from the trajectory calculations, in the rotating coordinate system and in the vicinity of the moon, is shown in figure 4. From such calculations, estimates were made for allowable errors in initial conditions to impact on some part of the moon.

Results of the impact error analysis are shown in figure 5. The velocity tolerance increases with an increase in injection velocity and then decreases; whereas the angular tolerances appear to approach an asymptote for injection velocities near the escape velocity. The angle β is the heading angle, and the angle ψ is related to firing time tolerance, a 1° change in ψ being approximately equal to 4 minutes in firing time. A comparison and discussion of the differences between the results obtained from two- and three-dimensional error analyses are given in a subsequent section.

Trajectories and Error Analysis for a Close Lunar Satellite

In the establishment of a close lunar satellite, it is necessary to place the space vehicle close to the moon, at which position the retro-rocket is fired to obtain a proper orbital velocity with respect to the moon. It is of interest to determine the effect of errors in the initial conditions on the accuracy of placing the lunar vehicle at a desired target point with respect to the moon.

The target point for this error analysis is chosen somewhat arbitrarily as a point approximately 5,000 miles below the center of the moon, measured perpendicular to the earth-moon plane. A lunar orbit

initiated approximately above or below the lunar poles¹ will result in a nearly polar orbit and insure complete reconnaissance coverage of the lunar surface over a period of one-half month.

Initial conditions were determined for placing the vehicle approximately 5,000 miles below the center of the moon. Variations were made in the initial conditions to study the resulting miss distances from the target point. Typical trajectories in the vicinity of the moon, in the rotating axis system, are shown in figures 6 to 8. These figures show how the various changes in initial injection conditions affect the trajectories and miss distances in the plane of the moon's travel and in the plane normal to this plane. The figures also show how the trajectories approach the moon more nearly normal to the direction of the moon's motion as the initial velocity is increased. The time of flight to the target point and the velocity with respect to the moon at the target point are shown in figure 9.

Some trajectories have been calculated for $\gamma = 17^\circ$ to give an indication of the effects of a different value of the injection angle. The basic conditions for these trajectories are also those for a due east injection from latitude $28^\circ 27.6'$ North (the latitude of Cape Canaveral, Florida) and with an initial velocity equal to the parabolic velocity. These trajectories are shown in figure 8. The main differences in changing the injection angle from 25° to 17° for these basic conditions are that the initial plane of motion of the space vehicle is more inclined to the earth-moon plane for the smaller injection angle (approximately 18° for $\gamma = 17^\circ$ as compared with 12° for $\gamma = 25^\circ$), and the orientation of the space-vehicle spin axis is modified. The effect of the greater inclination of the planes is seen particularly in the greater miss distances in the xz plane for equal velocity errors in figure 8 as compared with figure 7. Optimum values of the injection angle have not been obtained for this analysis and determination of the optimum values should be considered. Conflicting requirements on the best value for injection angle result from considerations of: allowable errors in launch conditions, velocities obtainable from the boost vehicle, aerodynamic heating, alinement of retro-rocket with respect to the velocity vector at time of retro-rocket firing, and so forth.

In order to provide a comparison between the present analysis and the impact-error analysis, estimates were made of the allowable errors in initial conditions for the space vehicle to penetrate a hypothetical sphere with radius equal to that of the moon but located with center approximately 5,000 miles below that of the moon. The results for this

¹For the purposes of this analysis, it is assumed that the lunar polar axis is normal to the earth-moon plane, whereas it is actually inclined to the normal by approximately 5° .

target-point error analysis are presented along with the results of the impact-error analysis in figure 5. The comparison indicates that more accuracy is required (about twice the angular accuracy) to hit within a spherical surface situated at this distance below the moon than is required to hit the moon. Some calculations for a target point approximately 3,000 miles below the center of the moon indicate that the accuracy required at this closer distance is approximately the same as that required at the greater distance. Combinations of errors in the initial conditions were investigated for several trajectories, and the miss distances were found to be directly additive for the range of errors investigated.

Lunar Satellite Orbit Considerations

It is of interest to determine the characteristics of lunar satellite orbits obtainable for some of the trajectories presented in figures 6 to 8. The characteristics of the lunar orbits obtainable and the amount of retro-rocket incremental velocity required will depend primarily on the distance of the vehicle from the moon, the retro-rocket orientation at the time of retro-rocket firing, and on the initial injection velocity. If the carrier rocket is assumed to be spin-stabilized at injection and the retro-rocket is assumed to be directed along the spin-axis, the retro-rocket orientation will remain fixed in inertial coordinates.

For this situation, values of the retro-rocket incremental velocity required to obtain polar orbits of the minimum eccentricity have been calculated by use of the two-body (moon-satellite) equations for the basic cases. These results are presented in figure 10. The retro-rocket is fired when the vehicle reaches the target point approximately 5,000 miles directly beneath the moon. For the basic cases of figures 6 to 8, the orientation of the retro-rocket thrust axis at the time of retro-rocket firing is indicated in the earth-moon plane and in the plane normal to the earth-moon plane by the lines with arrows. The apsides of the lunar orbits obtained with the indicated amount of retro-rocket velocity are shown in the bottom part of figure 10. For $\gamma = 25^\circ$, nearly circular orbits are obtained for launch velocities near parabolic velocity. The comparison of the apsides of the orbits for $\gamma = 25^\circ$ and $\gamma = 17^\circ$ at $\frac{V}{V_p} = 1.0$ shows that a relatively small change in the injection angle and the corresponding change in the orientation of the retro-rocket can have a considerable effect on the characteristics of the lunar orbit.

If the orientation of the retro-rocket were controllable so that the retro-rocket axis could be aligned with the velocity vector at the point of retro-rocket firing, less retro-rocket velocity increment would be required (as shown in the curve in fig. 10) and circular orbits could

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be obtained. Orientation control of the retro-rocket would, however, involve a more complicated system than the simple spin-stabilized system.

Calculations have been made to show the effects of errors in the initial launch conditions on the characteristics of the lunar orbits. These calculations were made for velocity ratios of 0.99247 and 1.000 for $\gamma = 25^\circ$, and 1.000 for $\gamma = 17^\circ$. The retro-rocket axis is assumed to be fixed in inertial coordinates at injection, and the retro-rocket is fired at a point along the trajectory (at approximately the closest approach to the moon) denoted by the ticks in figures 6 to 8. As mentioned previously, the orientation of the retro-rocket thrust axis for the basic cases of figures 6 to 8 is indicated by the lines with arrows.

The initial apsides of the lunar orbits and the orbital inclination with respect to the lunar polar axis are shown as functions of errors in the initial conditions in figures 11 to 13. These characteristics were calculated by use of two-body equations. For $\gamma = 25^\circ$, figures 11 and 12 show that reasonably close orbits can be achieved with tolerances in the initial conditions of the order of those required for lunar impact. For $\gamma = 17^\circ$, figure 13 indicates that some of the orbits obtained are probably not acceptable, largely because the orbit for the basic case has a large eccentricity. This figure simply illustrates that careful consideration must be given to the choice of injection angle, and thus retro-rocket orientation, for achieving satisfactory orbits. It is interesting to note from figures 8 and 13 that there is apparently a strong effect of the radius at which the retro-rocket is fired on the apsides of the orbits so that when the retro-rocket is fired at smaller radii than that of the original target point, more favorable orbits are obtained. This effect indicates that, if one is limited to a particular range of injection angle, it might be possible that adjustments in the radius at which the retro-rocket is to be fired and in the amount of retro-rocket velocity provided could produce satisfactory orbits.

In order to evaluate the use of two-body (moon-satellite) equations for calculation of the apsides of the orbits and to investigate stability of the orbits, three-body calculations were performed on the electronic computer for some of the cases shown in figures 11 to 13. The two-body and three-body calculations for the initial apsides of the orbits for the cases in figures 11 and 12 give practically identical results; thus, the use of two-body equations is justified for initial orbital calculations in the vicinity of the moon. For some of the cases shown in figure 13, where the upper apsis is large, the two-body calculations were not very good approximations, although they are still useful for indicating the orbits which might be unstable.

The three-body orbital calculations were run for real times corresponding to about 1 month to investigate the stability of the orbits. It can be demonstrated from energy considerations that, for the orbits

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investigated in figures 11 and 12, the satellite cannot escape from the vicinity of the moon. Although these orbits appear to be relatively stable, they do show a gradual increase in eccentricity with time; this increase indicates a possibility of eventual collision with the moon. It is not considered practical to perform machine computations for sufficient lengths of time to determine whether and when collisions would occur for these preliminary demonstration orbits, but such computations would be desirable for studies leading to an actual moon shot. Some of the orbits for which the initial apsides are shown in figure 13 are unstable (for instance, $\Delta V = -80$ and -40 ft/sec; $\Delta \gamma = 0.5^\circ$; $\Delta \beta = -0.5^\circ$; and $\Delta \psi = 0.4^\circ$), and the satellite either impacts on the moon or escapes from the vicinity of the moon after only one or two revolutions.

Effect on the Velocity Tolerance of Injection Angle and Inclination of Plane of Vehicle Trajectory to Earth-Moon Plane

Although the angular tolerances in initial conditions for a particular lunar mission approach an asymptote as the initial velocity is increased, the velocity tolerance reaches a maximum value for initial velocities near the parabolic velocity. These effects are demonstrated in the error analyses for the missions of lunar impact and of placing the space vehicle within a desired position with respect to the moon in figure 5. If, then, for any of several reasons it is desired to launch the vehicle with a velocity near the parabolic velocity, it is profitable to consider the range of injection velocities that might give the greatest velocity tolerance, the angular tolerances being essentially constant in this range. This section is primarily devoted to a discussion of tolerances in the initial injection velocity and the influence of several factors on this velocity tolerance.

Two-dimensional velocity tolerance curve and effect of injection angle.— The general nature of the variation in allowable error in the initial velocity for impact somewhere on the surface of the moon is to be considered as a function of the initial velocity. A simple two-body analysis has been helpful in defining this curve for the case in which the space vehicle remains always in the earth-moon plane (two-dimensional case). From the closed solutions for the classical restricted two-body problem (in this case, earth and space vehicle), one can calculate the time of flight to a radius from the earth corresponding to that of the moon, and the angular difference θ between the initial radius vector and the radius vector to the target point, measured in the plane of the satellite. The effect of changes in the initial launch velocity on the time of flight and on the angle θ can be determined. The change in position of the moon with respect to the original target point is related

to the change in time of flight, whereas the change in position of the space vehicle is related to the change in the angle θ . The difference between the angular change in position of the moon and the change in position of the vehicle, as a function of the change in initial velocity, is compared with the allowable angular difference for impact on the moon. This comparison gives a means for estimating the velocity tolerance for lunar impact.

The results of such a two-body, two-dimensional analysis are presented as the curves in figure 14, in which are plotted the allowable errors in initial velocity to impact on some part of the lunar surface, as a function of initial velocity, for two values of injection angle. Also shown in figure 14 are some points obtained from a three-body error analysis conducted in the same manner as the error analyses shown in figure 5. (All of these calculations are again for injection at a radius from center of earth of 4,259 miles and an earth-moon distance of 229,100 miles). The two-body calculations were devised to give a general indication of the nature of the velocity tolerance curve; comparison of these results with the three-body results indicates that the agreement is reasonably good.

Perhaps the most striking feature of the allowable error curves in figure 14 is the existence of the reflex points and the branches in the curves. The reflex points and branches result from the fact that the rate of change of θ with time of flight (both expressed in radian measure) increases from values less than 1 to values greater than 1 as the initial velocity increases. This condition simply means that increases in velocity above a sufficiently low value of basic initial velocity for which lunar impact is obtained cause the vehicle trajectories to move relatively more and more out in front of the moon until a certain

value of initial velocity is reached (at which $\frac{d\theta}{d \frac{T}{P/2\pi}} = 1.0$). Additional

increases in velocity cause the trajectories to again approach the moon, sweep across, and then pass off behind the moon. This situation gives the double-valued portions of the curves shown by the branches. At the reflex points, the trajectories move with respect to the moon up to the forward edge of the moon for increase in initial velocity; then, without passing off the front of the surface, additional increases in velocity cause the trajectories to move back toward the trailing edge and off behind the moon.

The practical significance of the reflex points is that for initial velocities near these points the maximum tolerances in injection velocity are obtained. The velocity tolerances can be very large. For instance, for $\gamma = 25^\circ$ and an injection velocity ratio to parabolic velocity of

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about 1.002, the three-body error analysis indicates allowable velocity errors for lunar impact of about 700 and -80 feet per second.

Some other interesting features are illustrated by the curves of figure 14. The effect of injection angle is shown by a comparison of the upper and lower sets of curves. For a large injection angle, $\gamma = 25^\circ$, the overall allowable errors are greater than for $\gamma = 0^\circ$. However, the velocity for maximum allowable errors, that is, the reflex point, occurs at higher values of initial velocity as the injection angle is increased. (This trend has also been checked at an intermediate value of injection angle.) Thus, for injection at parabolic velocity, the allowable velocity errors for impact are 270 and -100 feet per second for $\gamma = 0^\circ$, but only 110 and -60 feet per second for $\gamma = 25^\circ$. It can be seen, therefore, that results from error analyses made at isolated values of injection velocity and injection angle can be misleading. Reference 4 presents a curve of allowable velocity error for impact as a function of initial velocity for the two-dimensional case which shows an increase near parabolic velocity and a decrease at velocities greater than parabolic, but the curve did not contain the discontinuities shown in figure 14.

Another point illustrated by figure 14 is that, for trajectories aimed at the center of the moon, equal values of positive and negative velocity tolerance are obtained only for injection velocities considerably higher than the velocity at the reflex point, and the overall tolerance for this condition is not the maximum. However, this situation can be corrected by aiming not at the center of the moon or other target point, but somewhat in front of or behind it.

Effect of inclination of plane of trajectory to earth-moon plane.-
The discussion in the previous section applies to trajectories which always remain in the earth-moon plane. For space vehicle launchings from latitudes greater than the maximum declination of the moon, it is no longer possible to launch the vehicle in the earth-moon plane without corrective guidance considerations, and the inclination of the vehicle trajectory to the earth-moon plane has a considerable effect on the velocity tolerance for lunar impact. Figure 15 shows a comparison of the results from the impact error analyses for injection from latitude $28^\circ 27.6'$ North and for injection in the earth-moon plane with data from figures 5 and 14. Because of the inclination of the planes, the results for injection from a moderate latitude no longer have double values or reflex points, and the allowable velocity tolerance for achieving lunar impact is considerably less than that from the two-dimensional analysis. It is again seen that results from two-dimensional analyses can be somewhat misleading if applied to vehicle injections from a moderate latitude.

With a modification of the simple two-body analysis described above to include the effect of inclination of the plane of the vehicle trajectory

to the earth-moon plane, some observations can be made concerning the overall effect of this inclination. The results of such an analysis are presented in figure 16 for a range of inclinations from vehicle trajectory in the plane of the moon to trajectory inclined 30° to the plane of the moon. The comparison shows that, even for small angles of inclination between the planes, the velocity tolerance is greatly reduced at certain values of initial velocity as compared with the tolerance for the two-dimensional case.

Some indication of the effect of initial latitude can be determined from figure 16. With the maximum declination of the moon of about 18.2° and with a due east heading, $\eta = 11.3^\circ$ corresponds to injection from latitude of about 28.5° and $\eta = 30^\circ$ from latitude of about 45° . (The minimum values of η for these latitudes are 10.3° and 26.8° , respectively.) The allowable tolerance in initial velocity for lunar impact for injection from the higher latitude is about half of that for injection from the lower latitude.

The two-body analysis, including consideration of the inclination of the plane of the vehicle trajectory to the earth-moon plane, is also useful in estimating the allowable velocity and injection angle for hitting within the sphere with radius equal to that of the moon but located approximately 5,000 miles below the center of the moon. The curves in figure 17 are for the two-body results and the circle symbols are results obtained from the three-body analysis. For this mission, the gravitational influence of the moon is relatively constant throughout the hypothetical target sphere, and the two- and three-body calculations show good agreement.

CONCLUDING REMARKS

From an analysis of three-dimensional lunar trajectories, calculated with use of the equations of motion of the classical restricted three-body problem of celestial mechanics and consideration of some lunar exploration missions, some general remarks can be made.

The allowable errors (particularly in the velocity tolerance) in initial conditions for achieving lunar impact or for hitting within a prescribed region with respect to the moon are strongly dependent on the values of initial injection velocity and injection angle. Also, there can be large differences in results obtained from two-dimensional analyses (in which the vehicle trajectory is assumed to remain always in the earth-moon plane) and those obtained from three-dimensional analyses, even for relatively small inclination angles between the plane of the vehicle trajectory and the earth-moon plane. It has been found that some of these accuracy tolerances can be fairly well estimated by use of a two-body

analysis which considers the inclination of the plane of the vehicle to the earth-moon plane.

The accuracy required to hit within a hypothetical sphere with radius equal to the moon radius, but with center a few thousand miles from the center of the moon, is greater than that required to hit the moon. However, satisfactory orbits for a relatively close lunar satellite can be obtained with accuracies in initial conditions approximately equal to those required for lunar impact. When the space vehicle is spin-stabilized at injection, careful consideration must be given to the choice of injection angle (and thus retro-rocket orientation) in order to achieve satisfactory lunar orbits.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., April 27, 1959.

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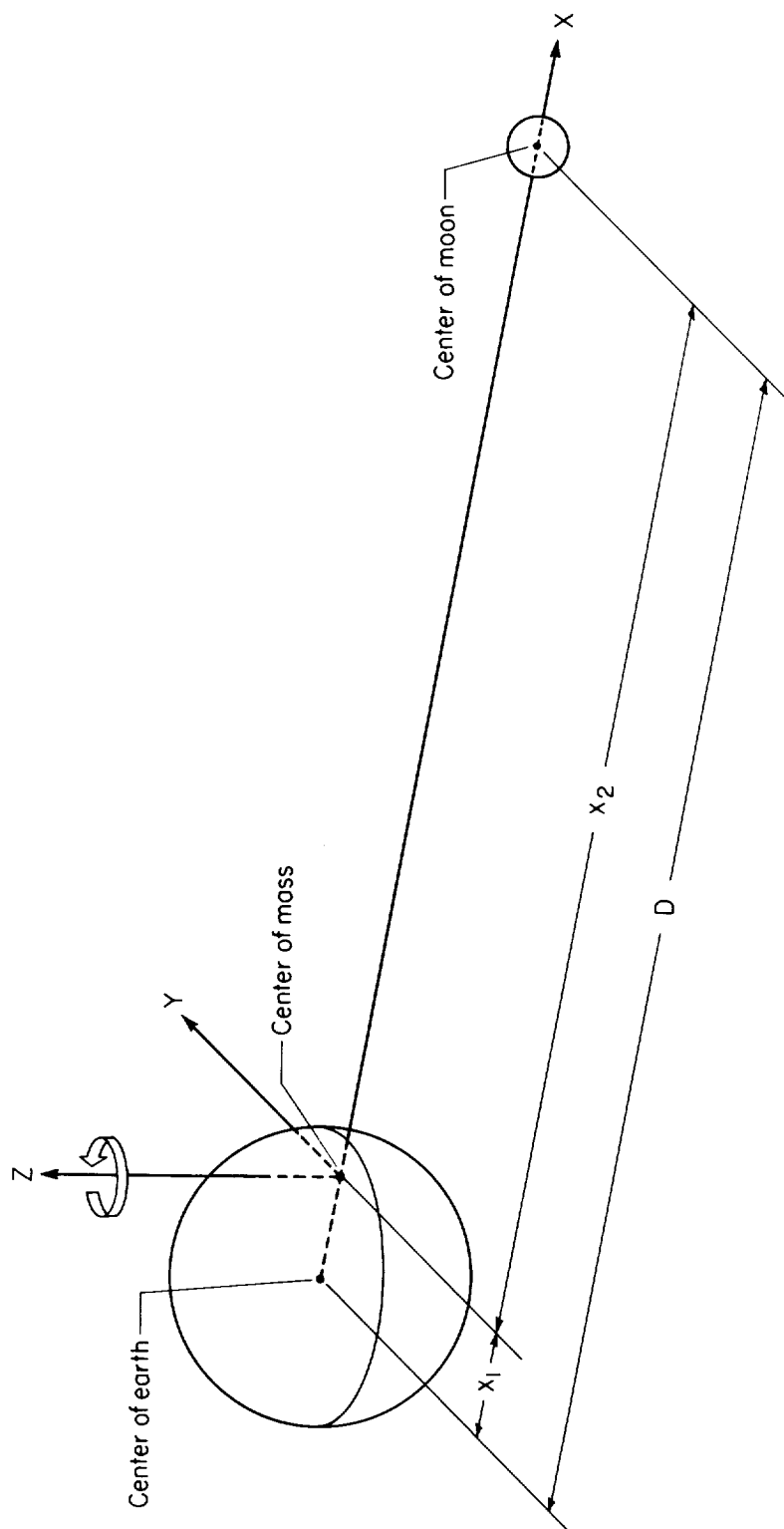


Figure 1.- Rotating coordinate system.

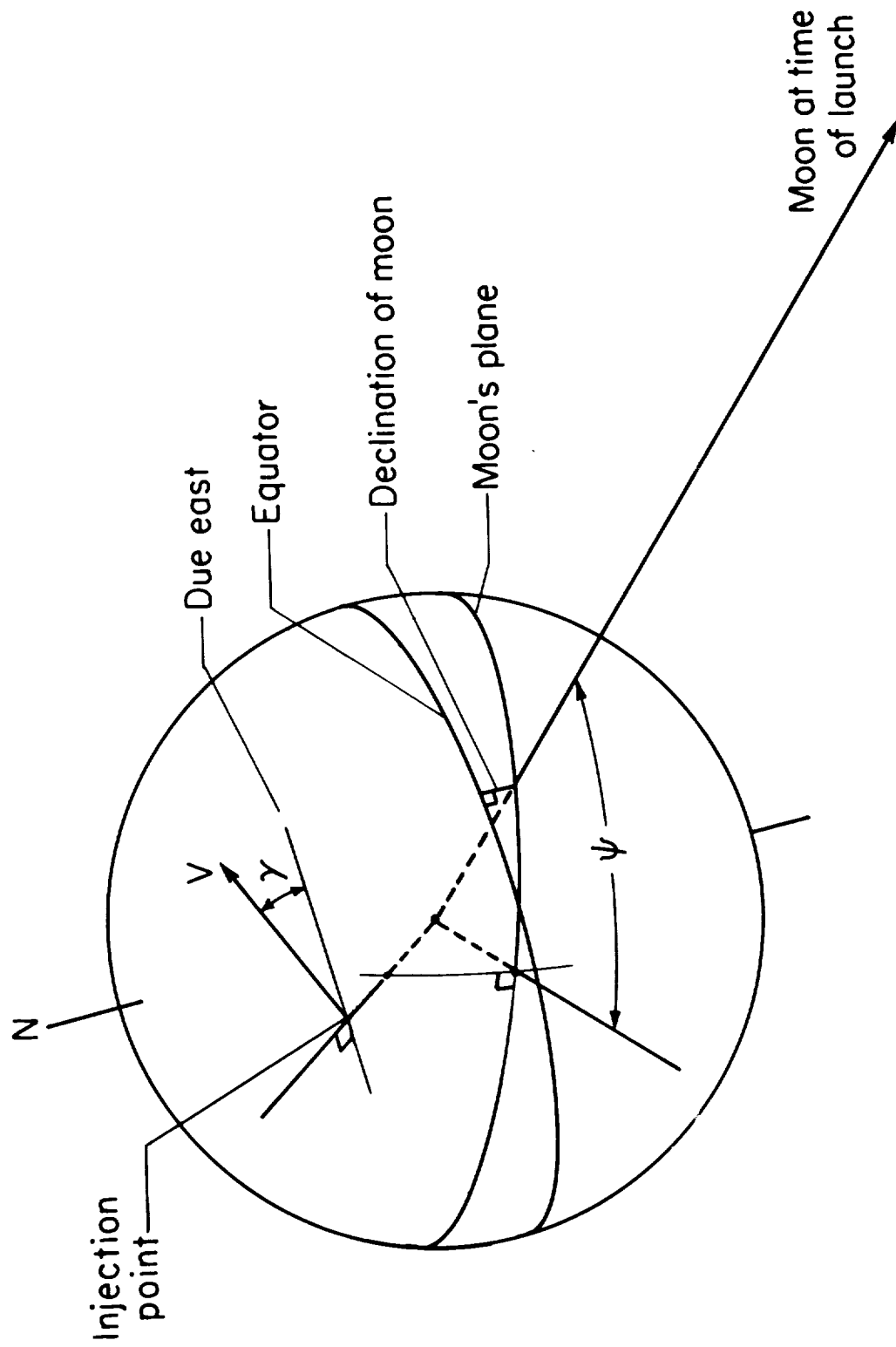


Figure 2.- Illustration of some parameters.

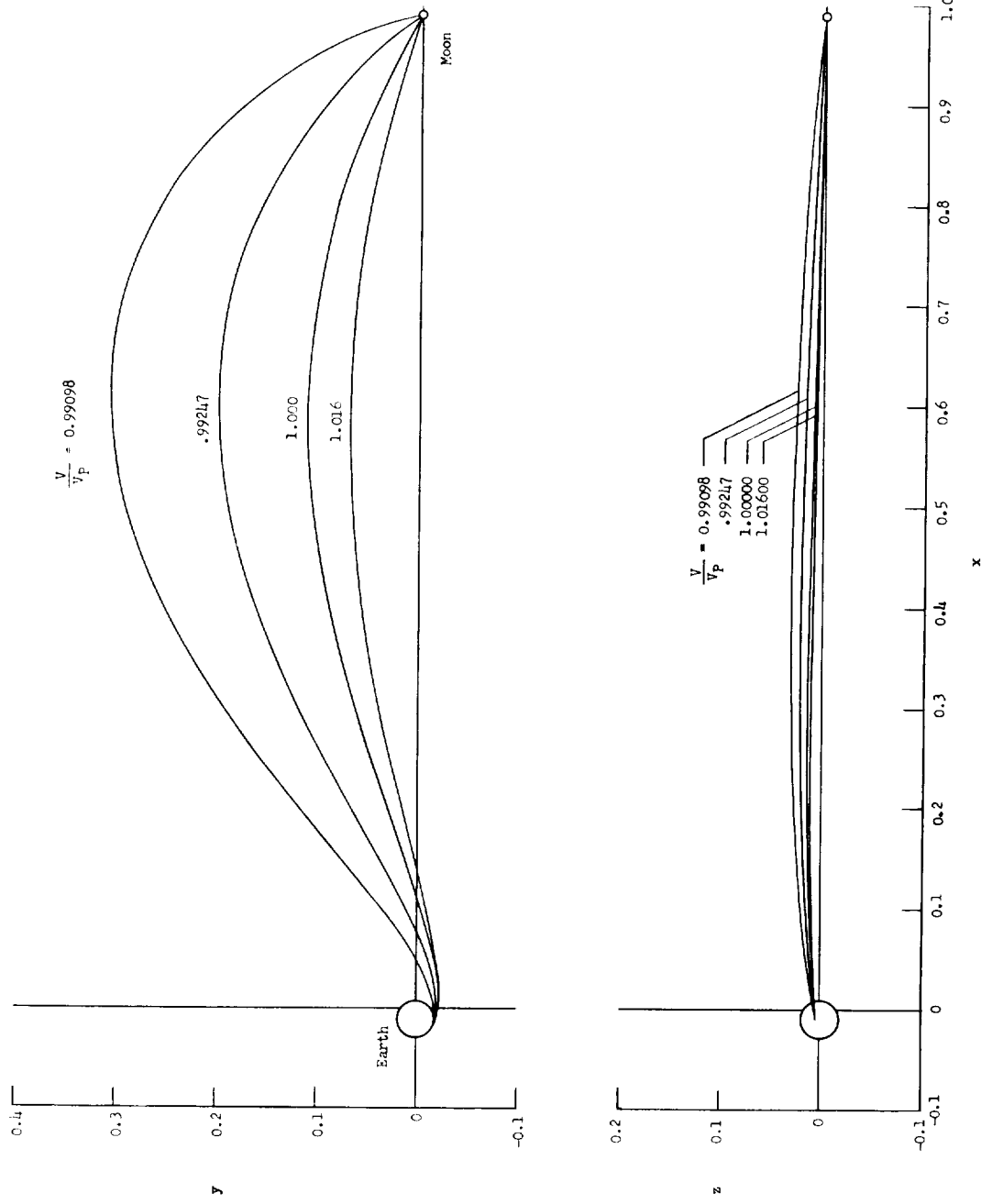
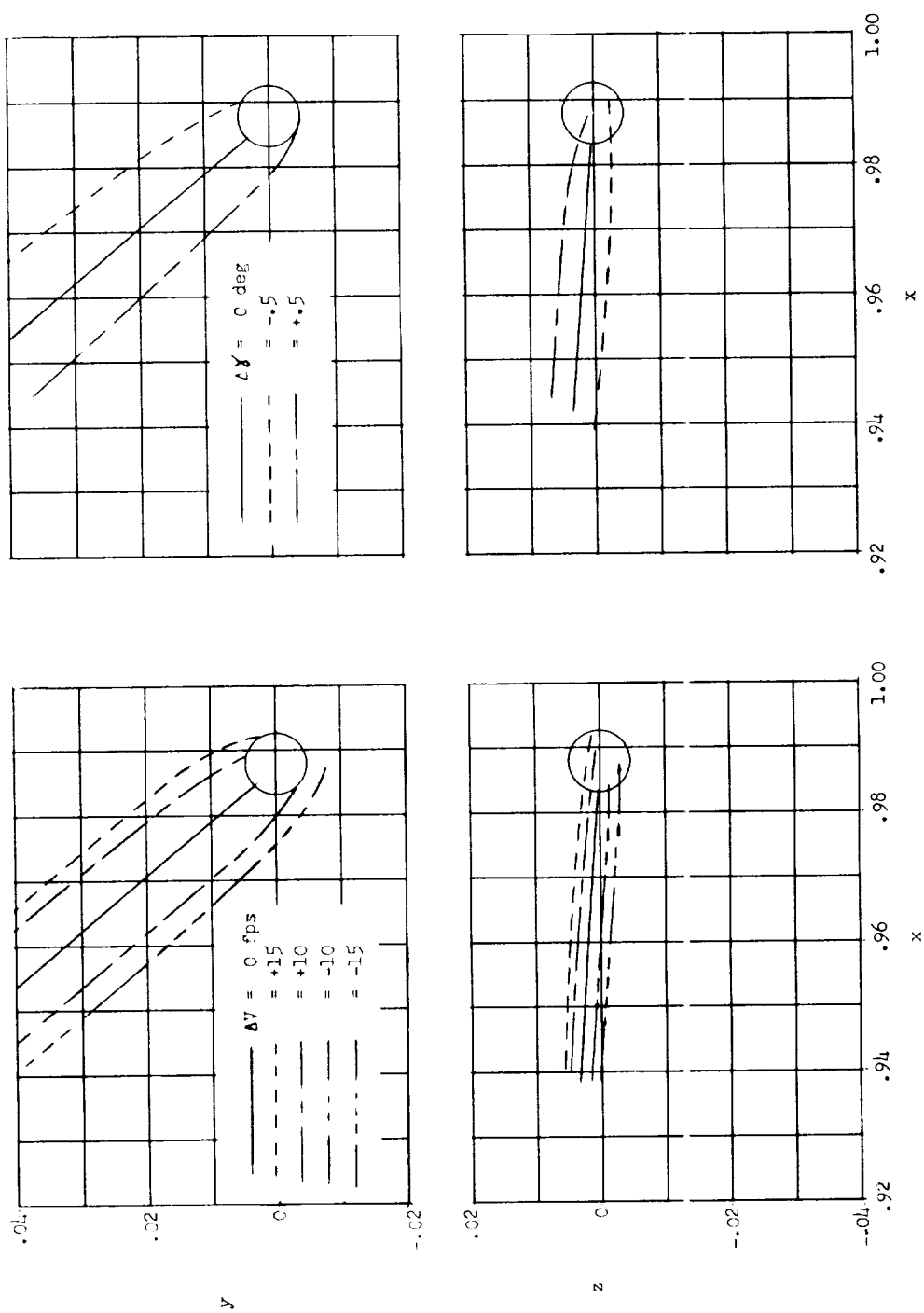
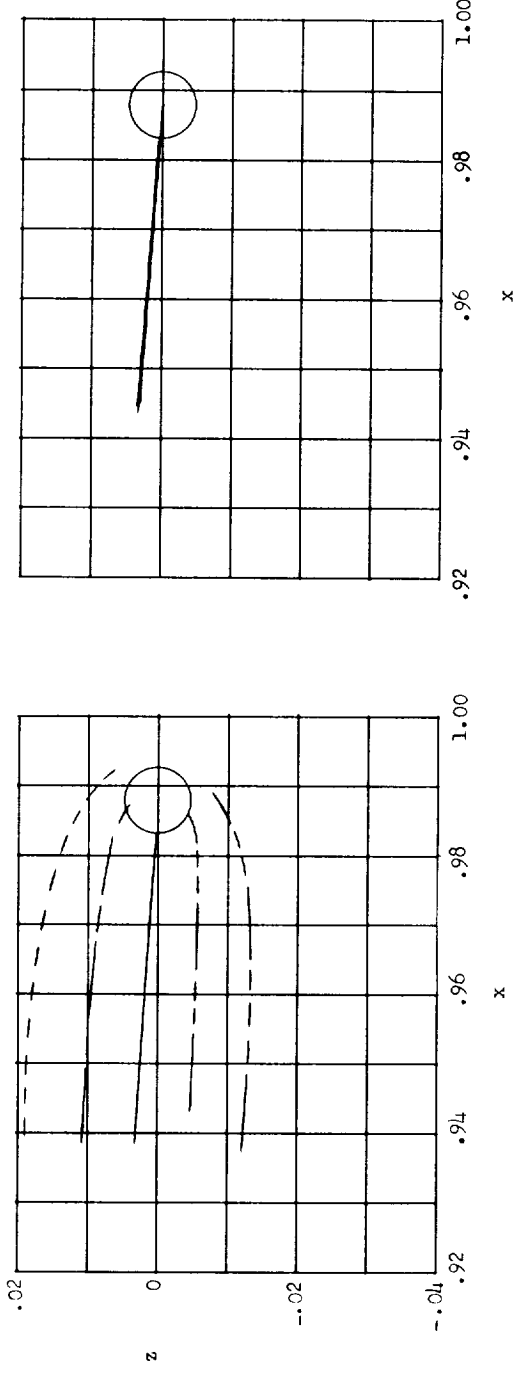
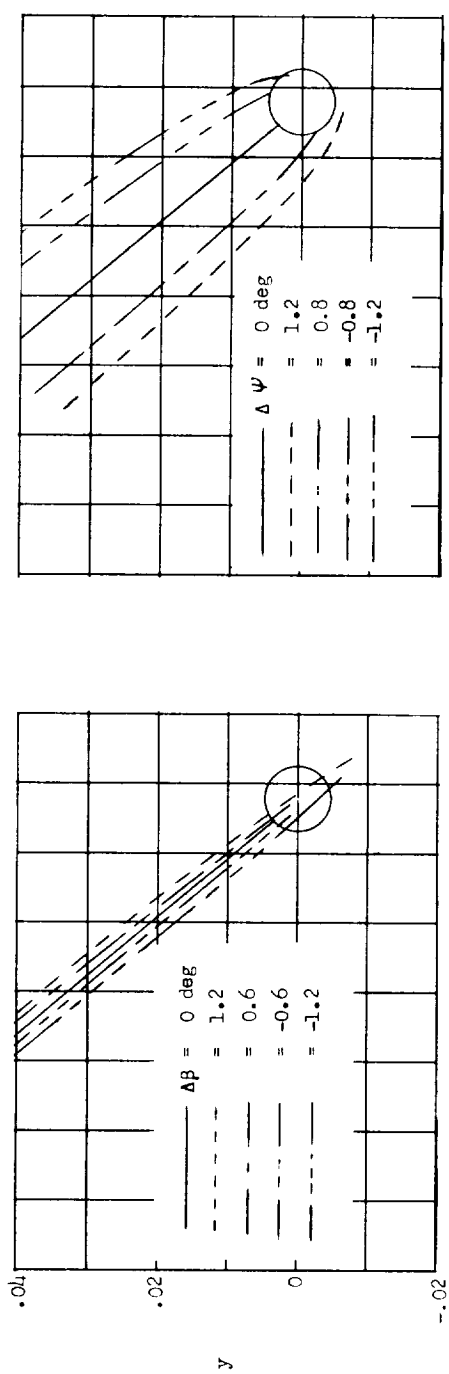


Figure 3.- Typical three-dimensional lunar impact trajectories; rotating axis system.



(a) Velocity errors. (b) Injection-angle errors.

Figure 4.- Typical trajectories for impact error analysis. $V/V_p = 0.99247$; $\gamma = 25^\circ$.



(c) Heading-angle errors.

(d) Position-angle errors.

Figure 4.- Concluded.

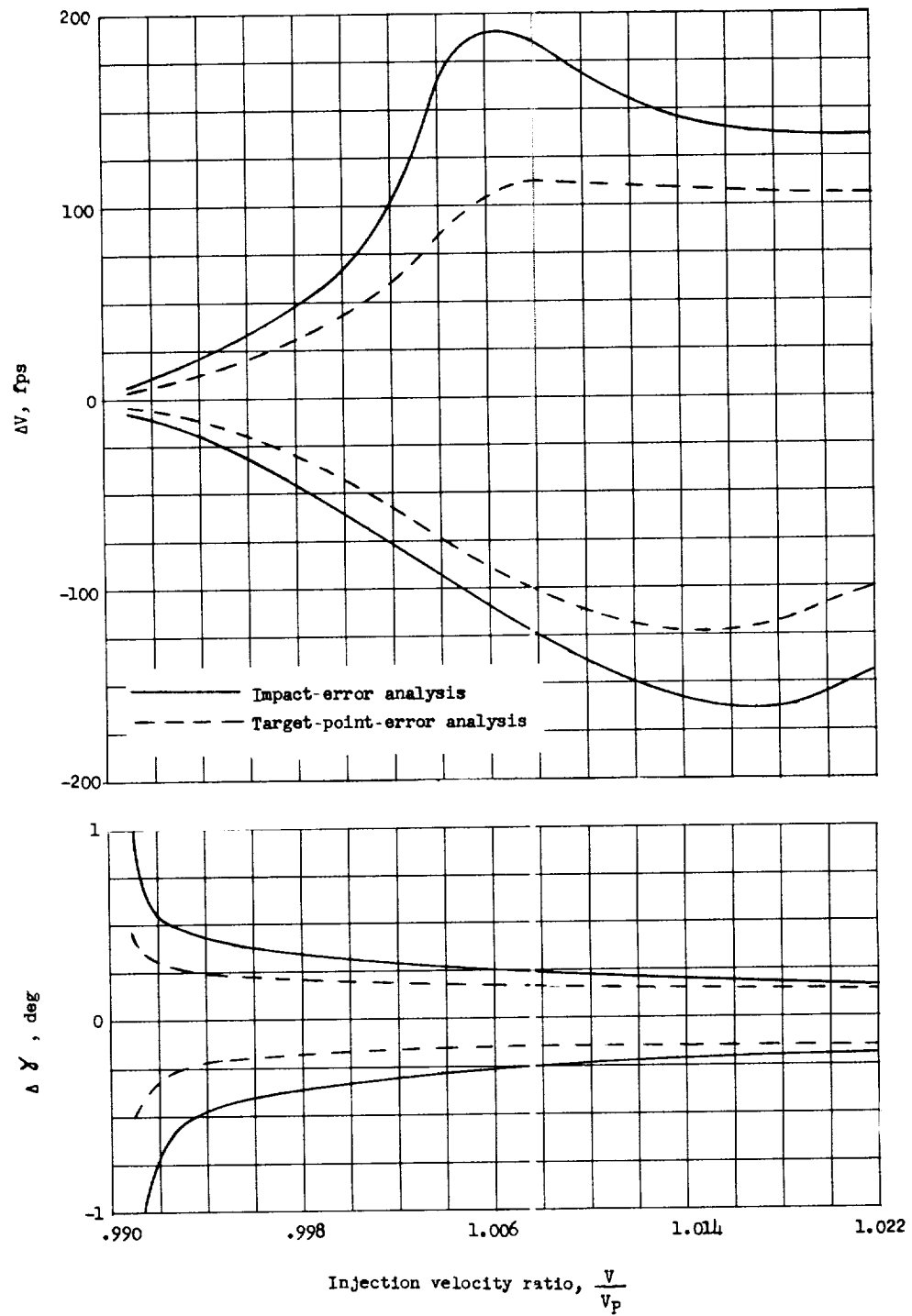


Figure 5.- Results of three-dimensional error analysis.

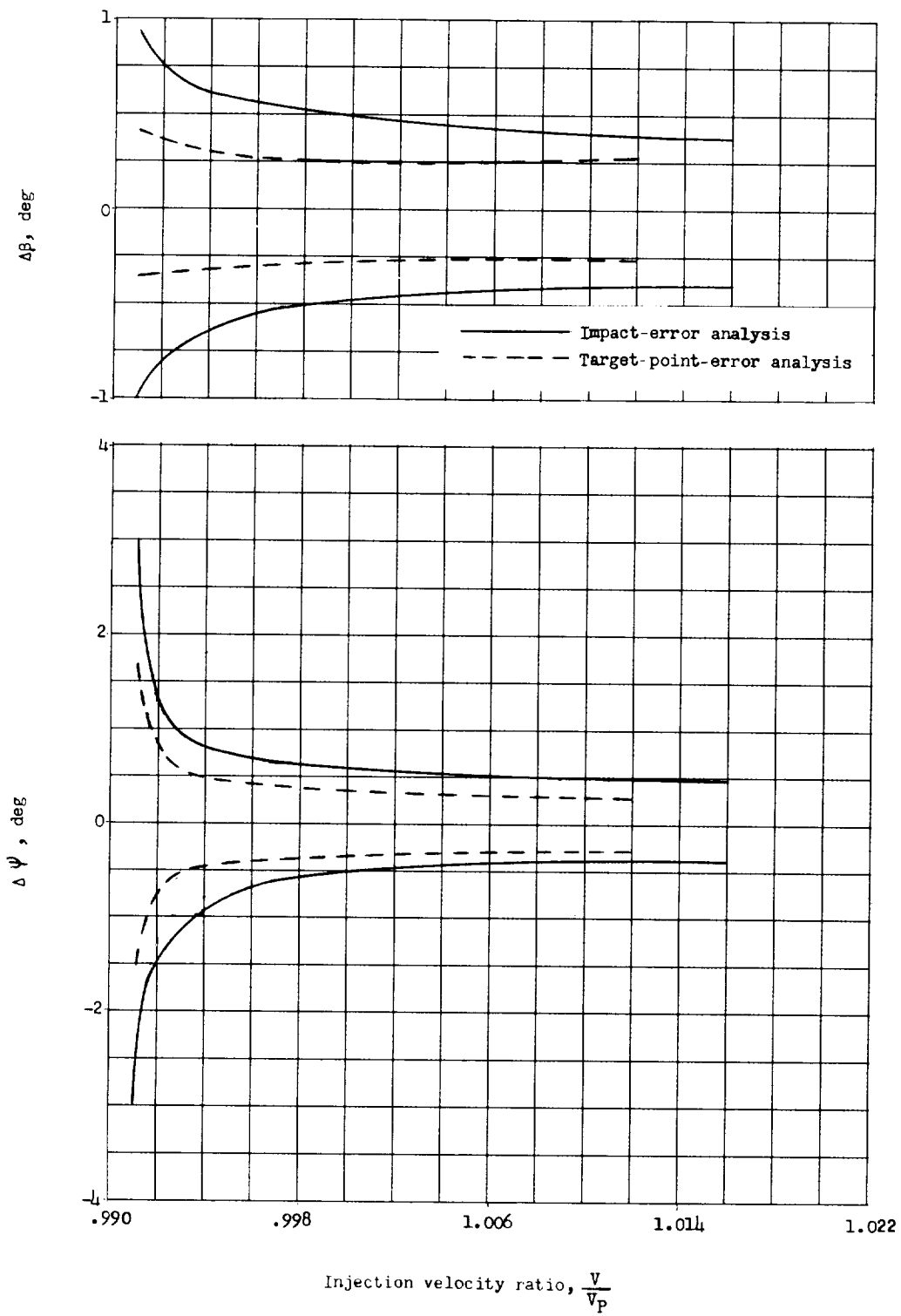
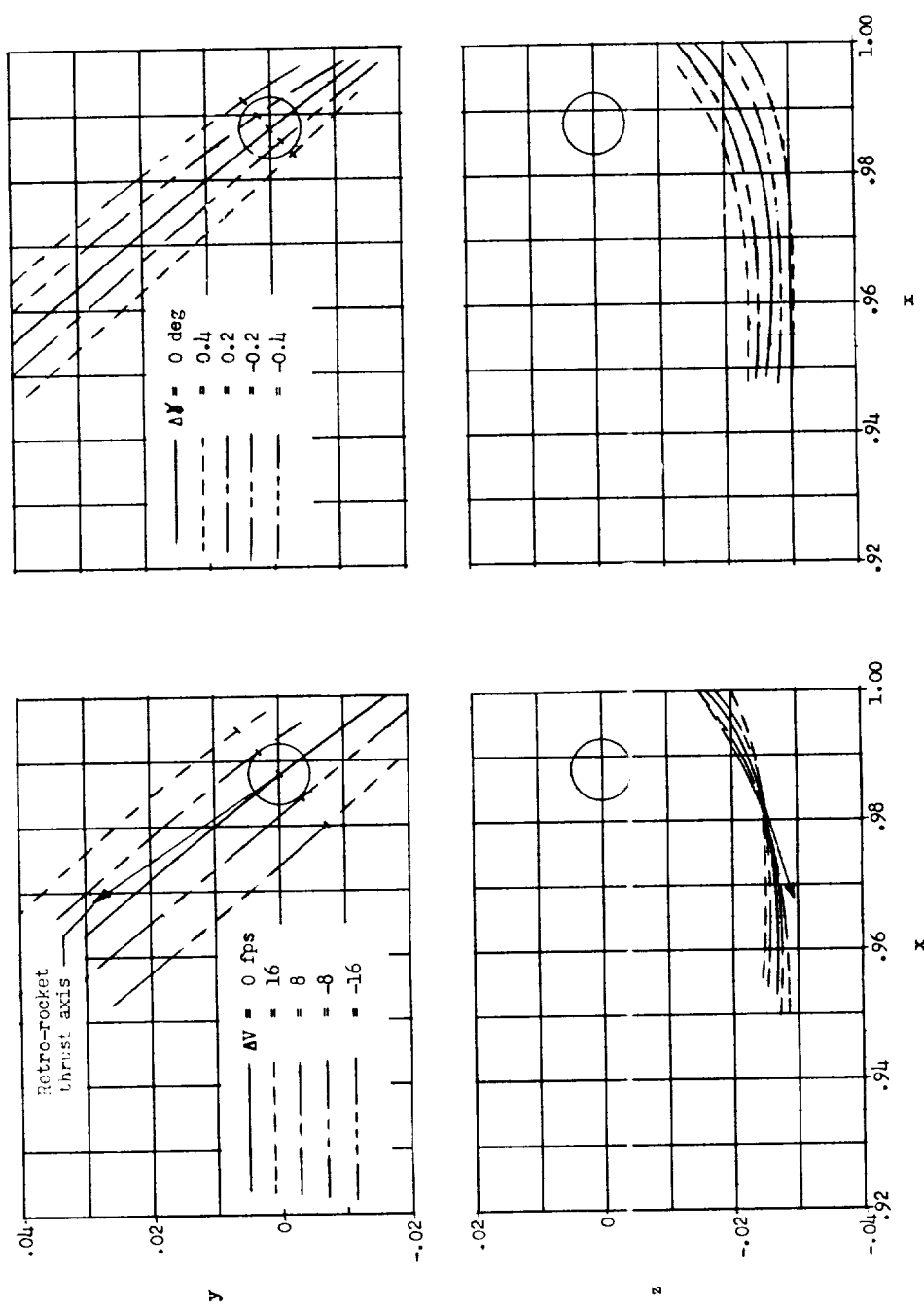


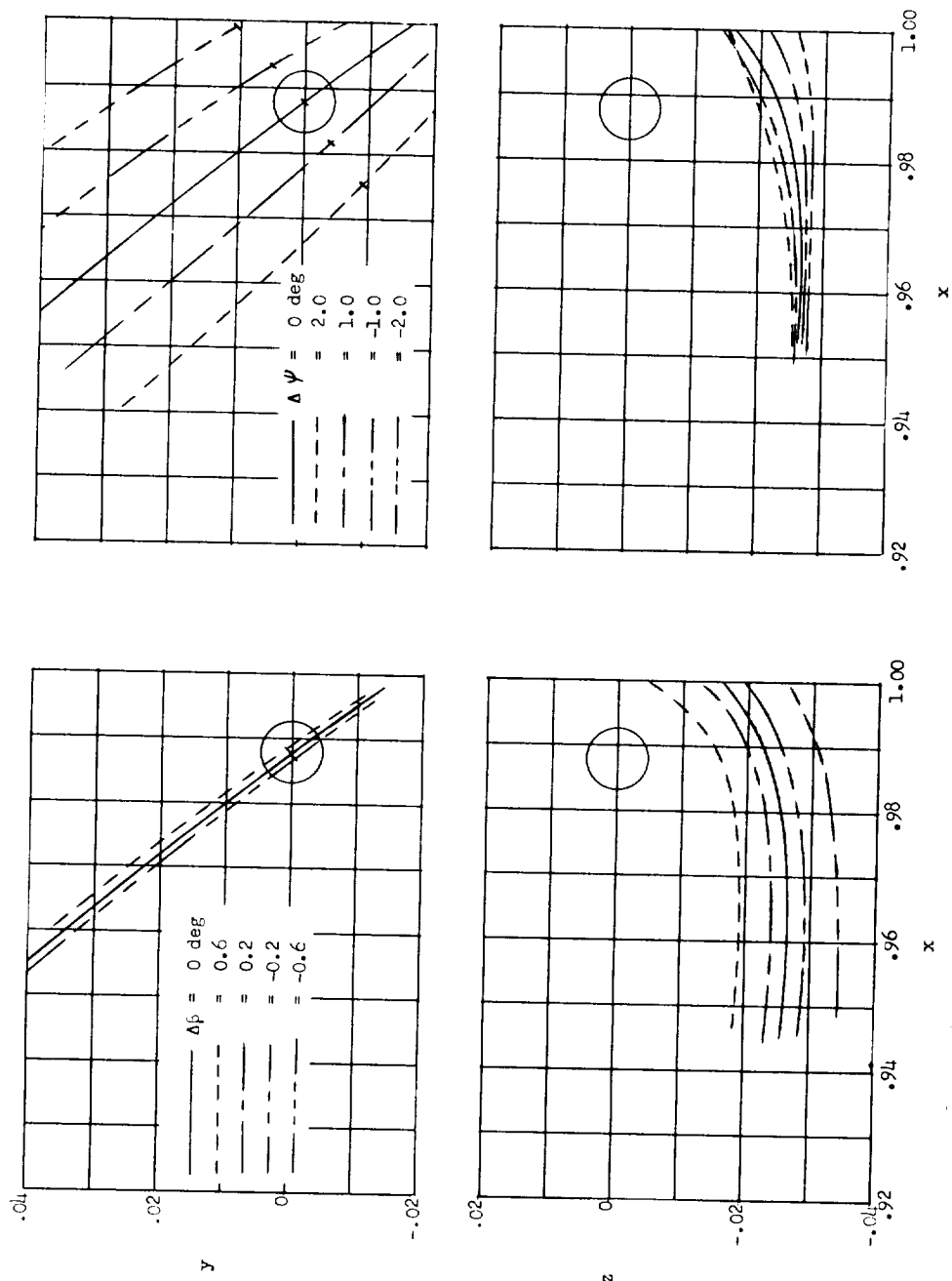
Figure 5.- Concluded.



(a) Velocity errors.

(b) Injection-angle errors.

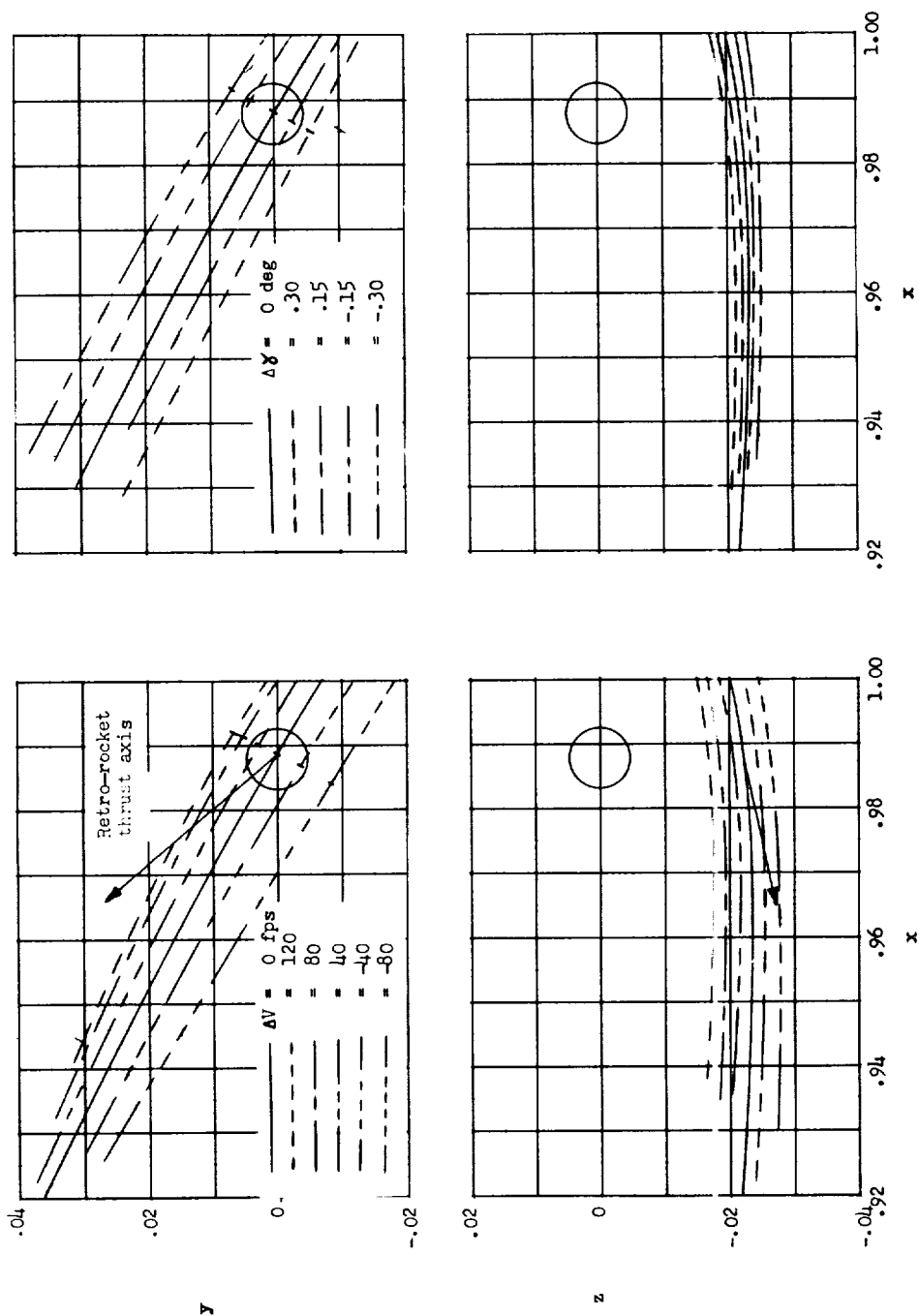
Figure 6.- Typical trajectory for establishment of a close lunar satellite.
 $V/N_p = 0.99247$; $\gamma = 25^\circ$.



(c) Heading-angle errors.

(d) Position-angle errors.

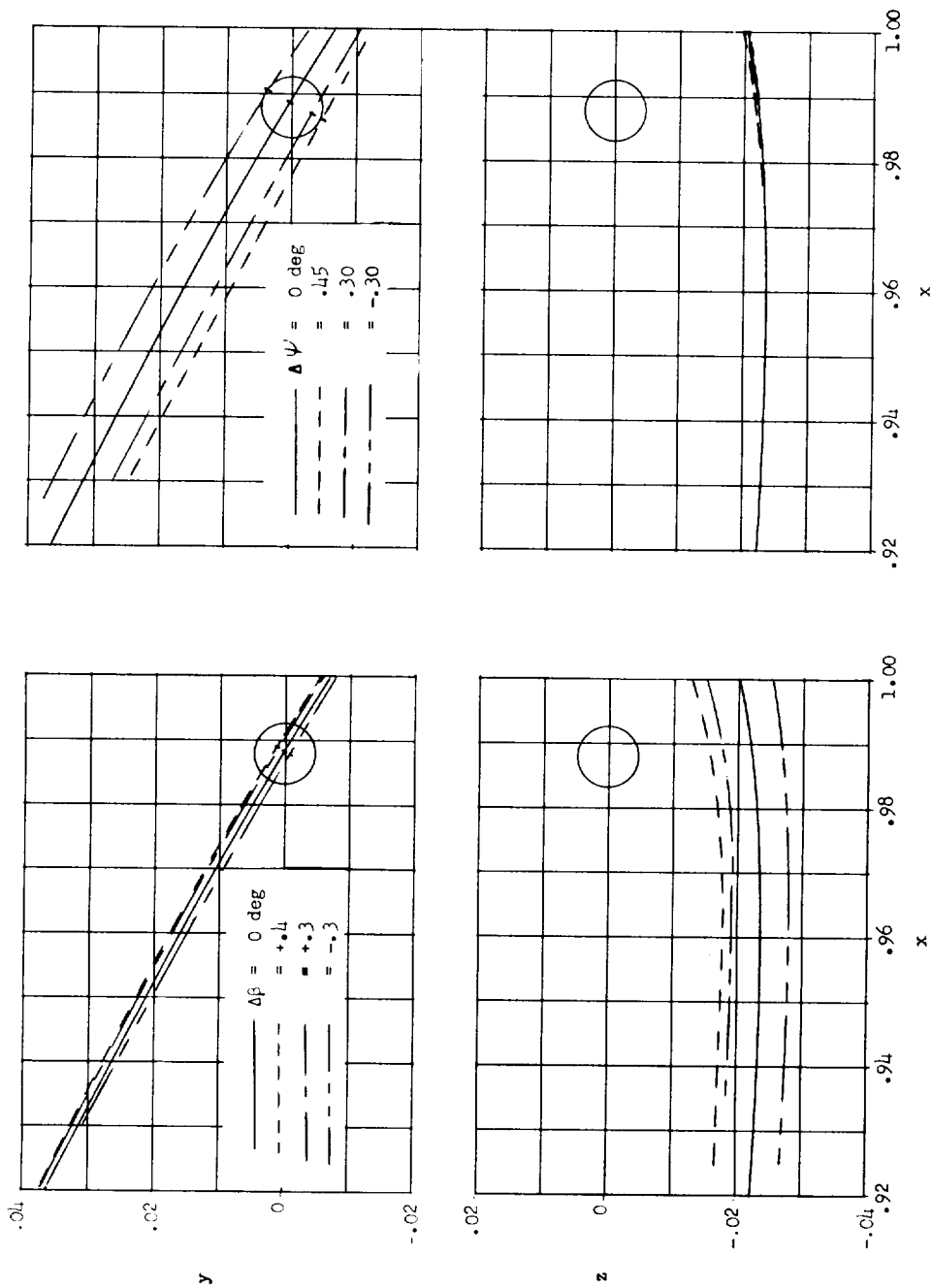
Figure 6.- Concluded.



(a) Velocity errors.

(b) Injection-angle errors.

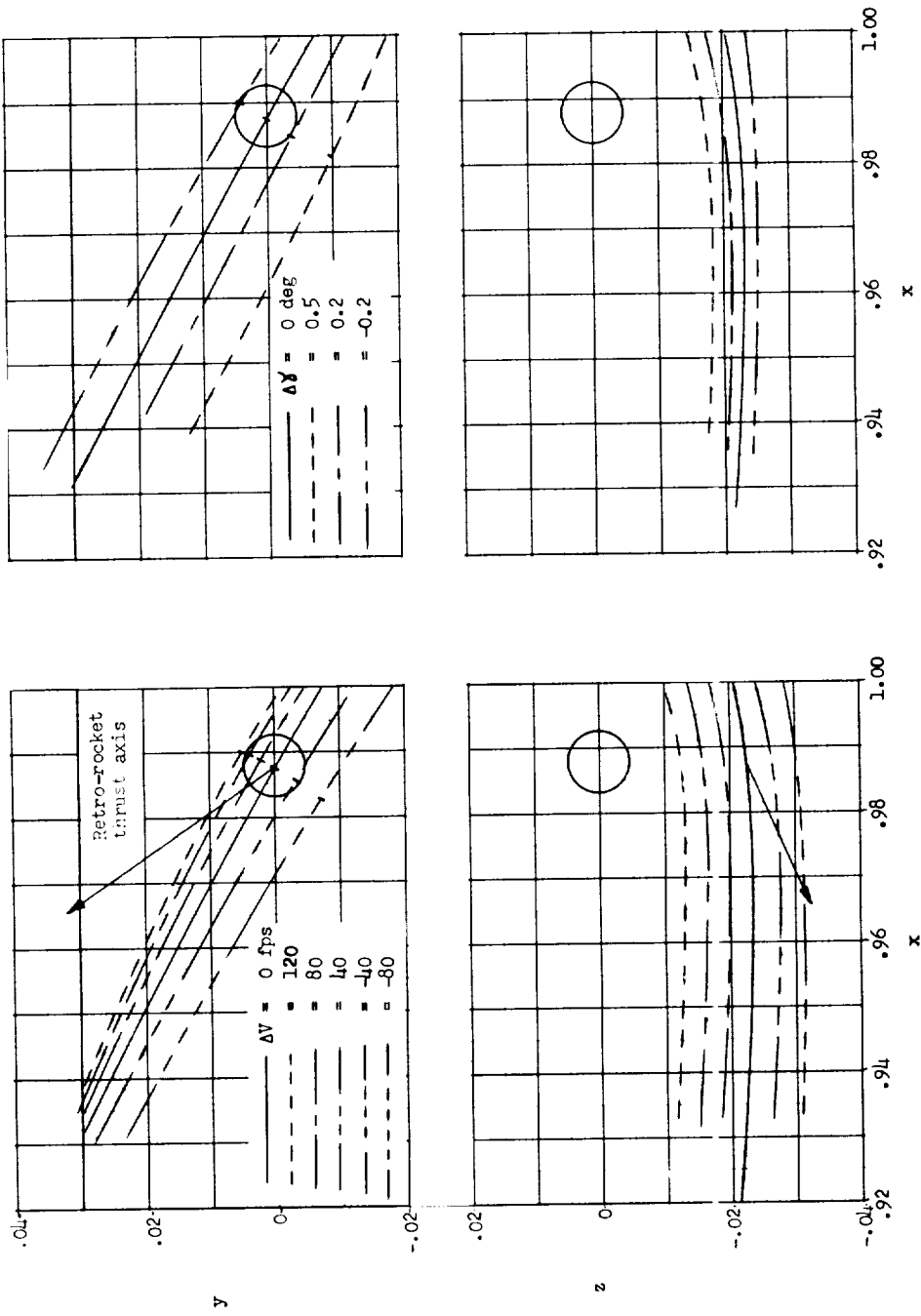
Figure 7.- Typical trajectory for establishment of a close lunar satellite.
 $V/V_P = 1.000; \gamma = 25^\circ$.



(c) Heading-angle errors.

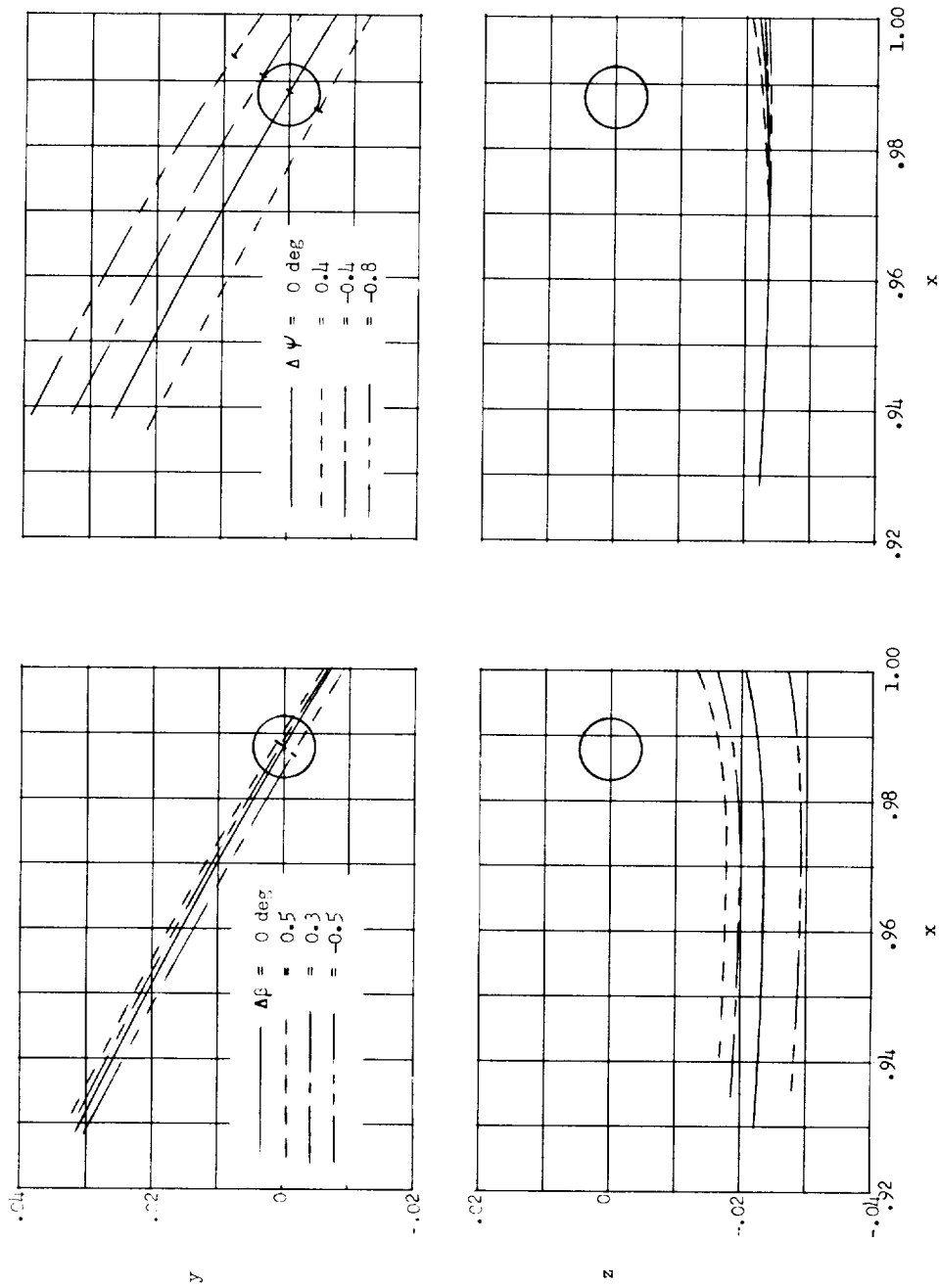
(d) Position-angle errors.

Figure 7.- Concluded.



(a) Velocity errors. (b) Injection-angle errors.

Figure 8.- Typical trajectory for establishment of a close lunar satellite.
 $V/V_P = 1.000$; $\gamma = 170^\circ$.



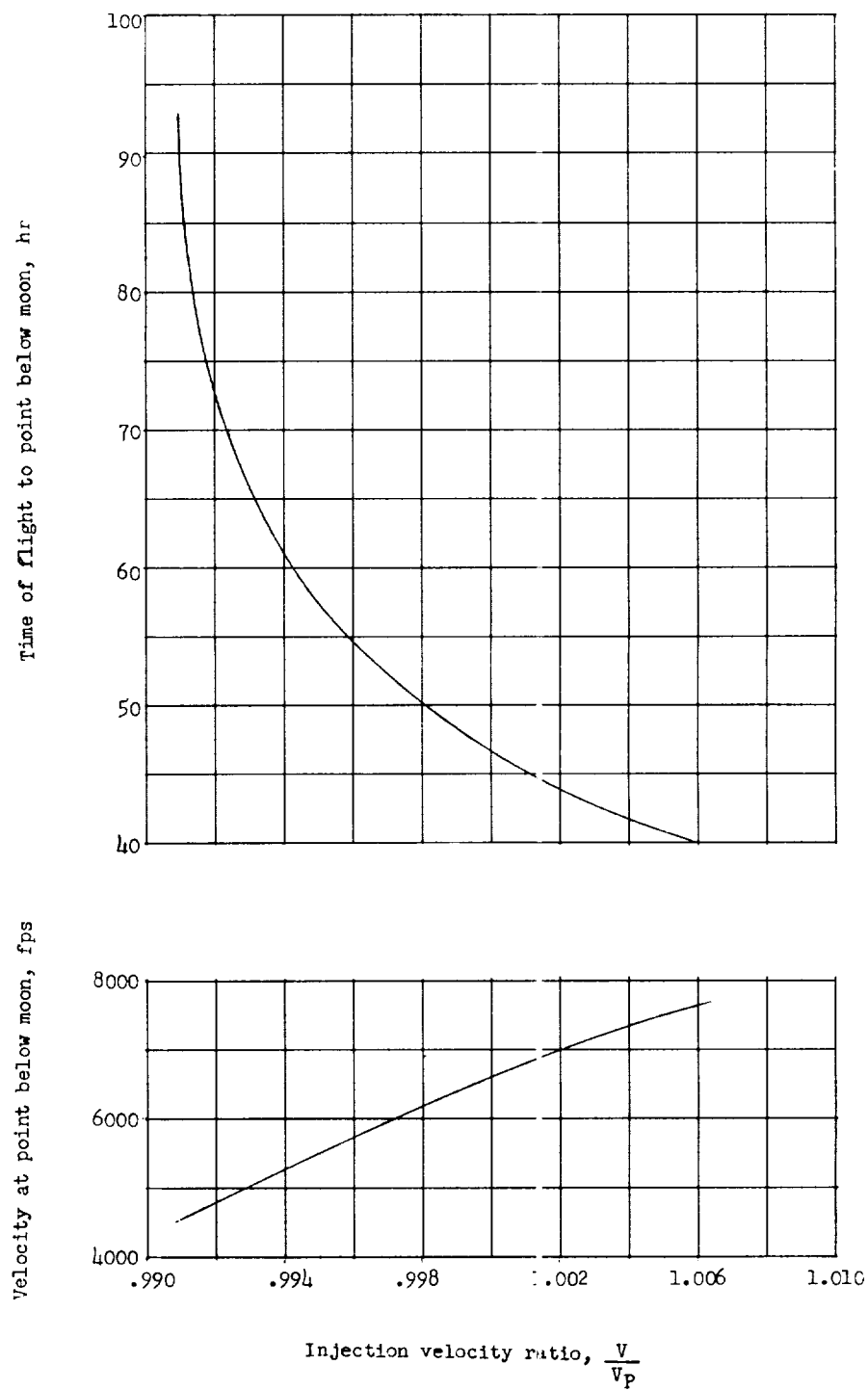


Figure 9.- Time of flight to point below center of moon and velocity at that point. $\gamma = 25^\circ$.

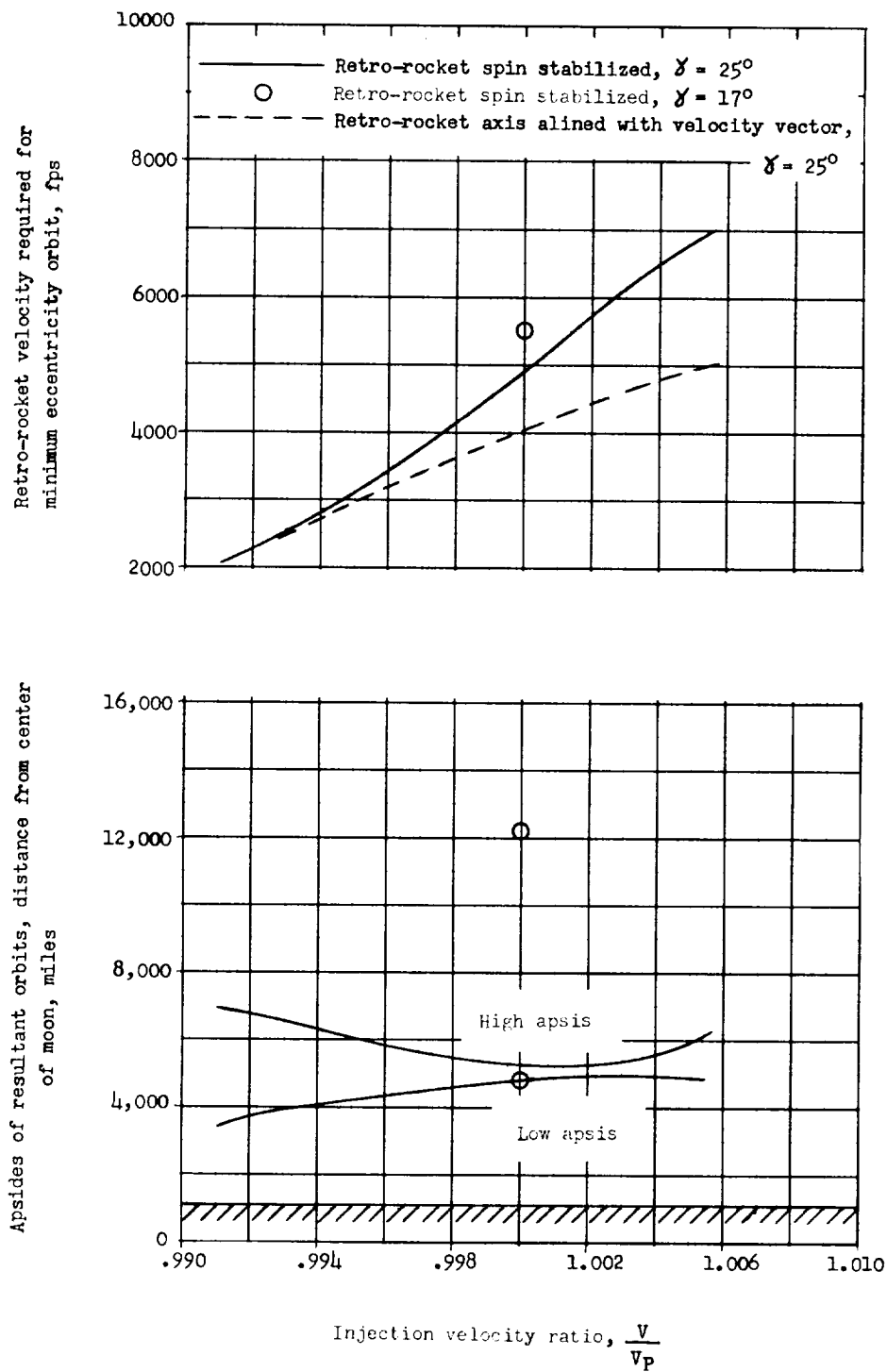


Figure 10.- Retro-rocket velocities required for minimum eccentricity orbits and apsides of resultant orbits.

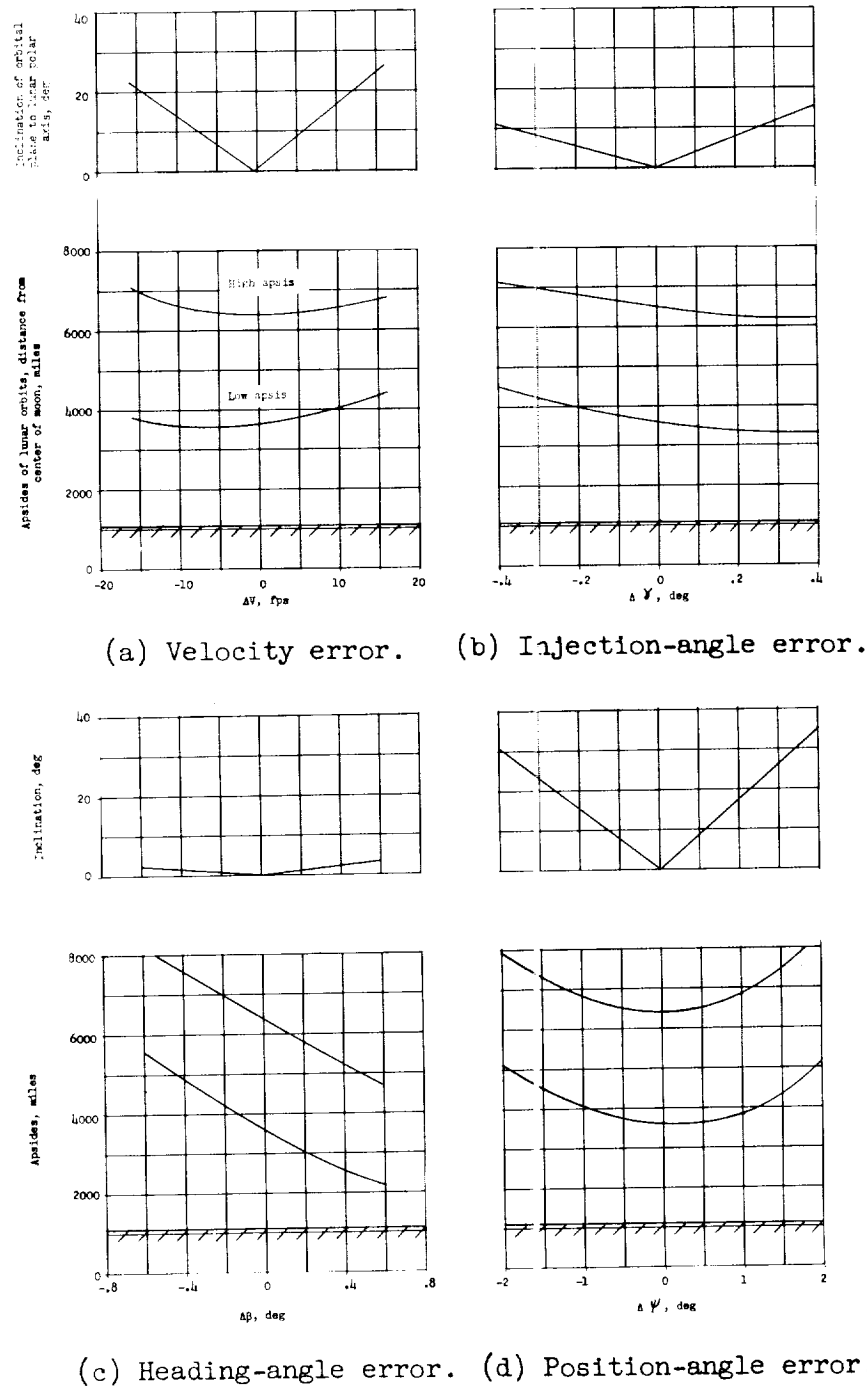
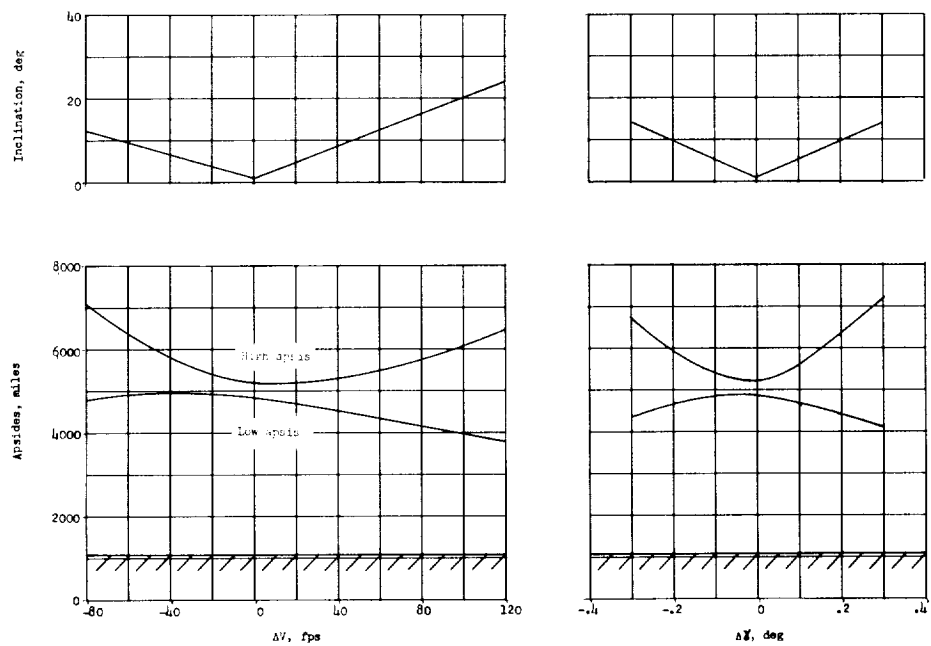
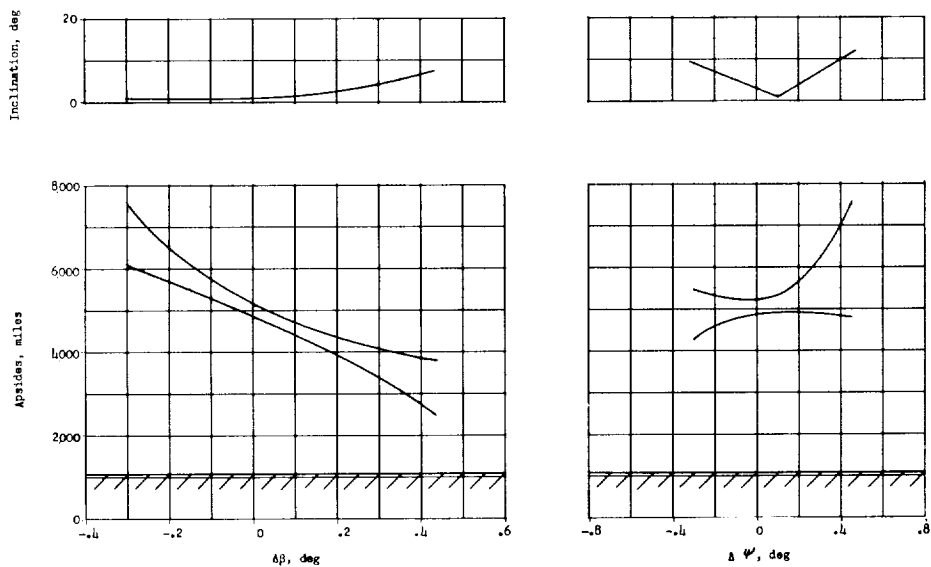


Figure 11.- Effect of errors in initial conditions on characteristics of lunar orbit. $V/V_p = 0.99247$; $\gamma = 25^\circ$.



(a) Velocity error.

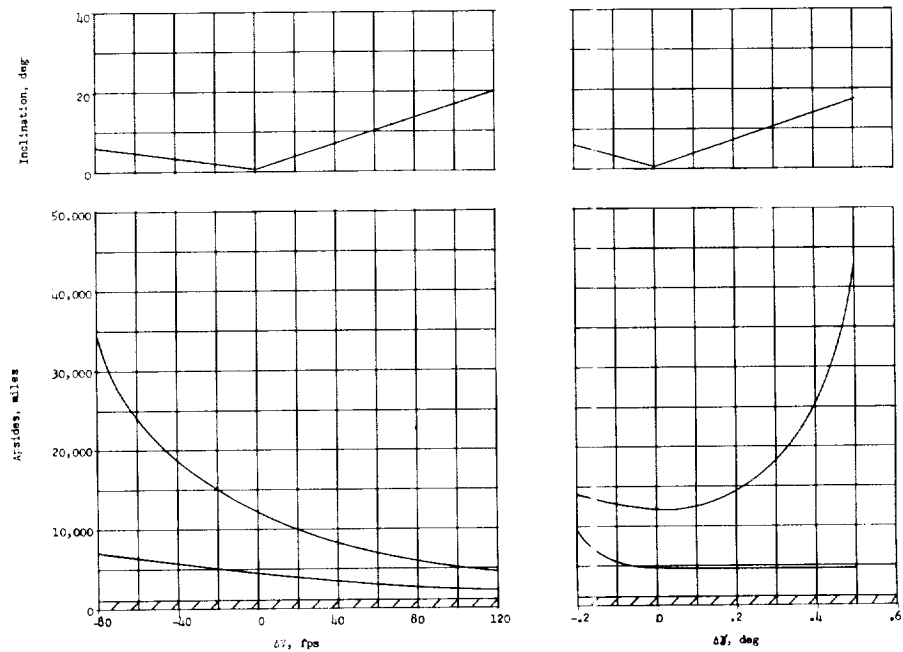
(b) Injection-angle error.



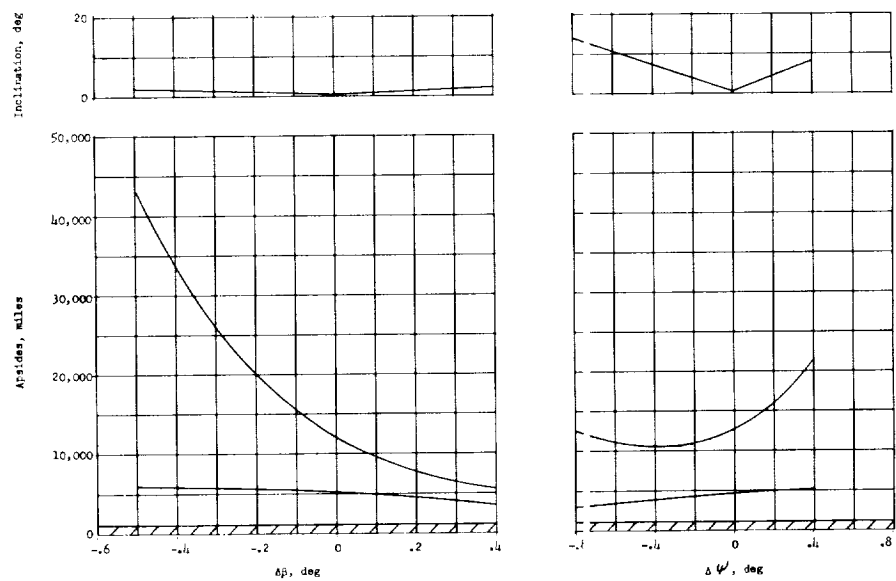
(c) Heading-angle error.

(d) Position-angle error.

Figure 12.- Effect of errors in initial conditions on characteristics of lunar orbit. $V/V_P = 1.000$; $\gamma = 25^\circ$.



(a) Velocity error. (b) Injection-angle error.



(c) Heading-angle error. (d) Position-angle error.

Figure 13.- Effect of errors in initial conditions on characteristics of lunar orbit. $V/V_p = 1.000$; $\gamma = 17^\circ$.

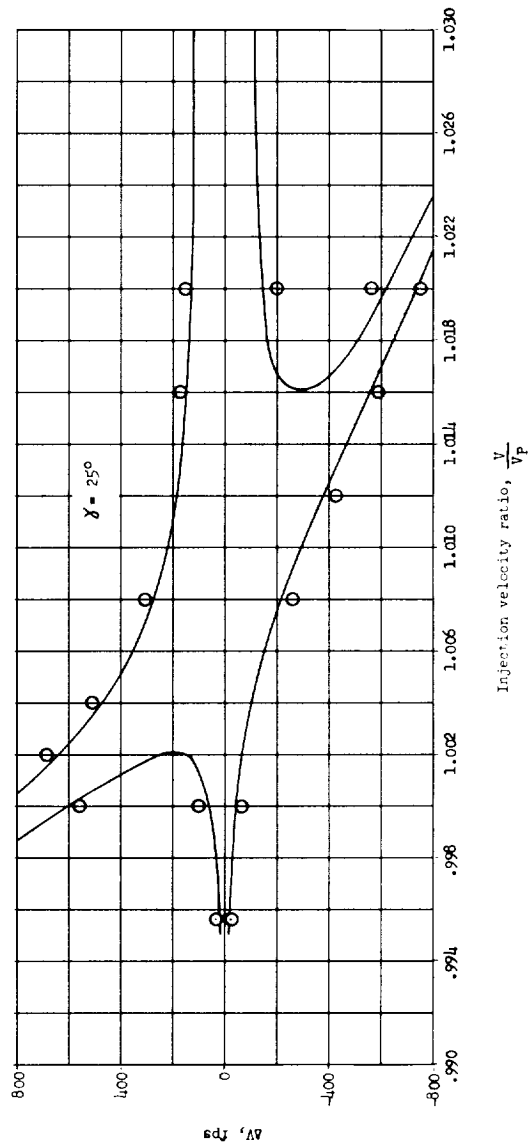
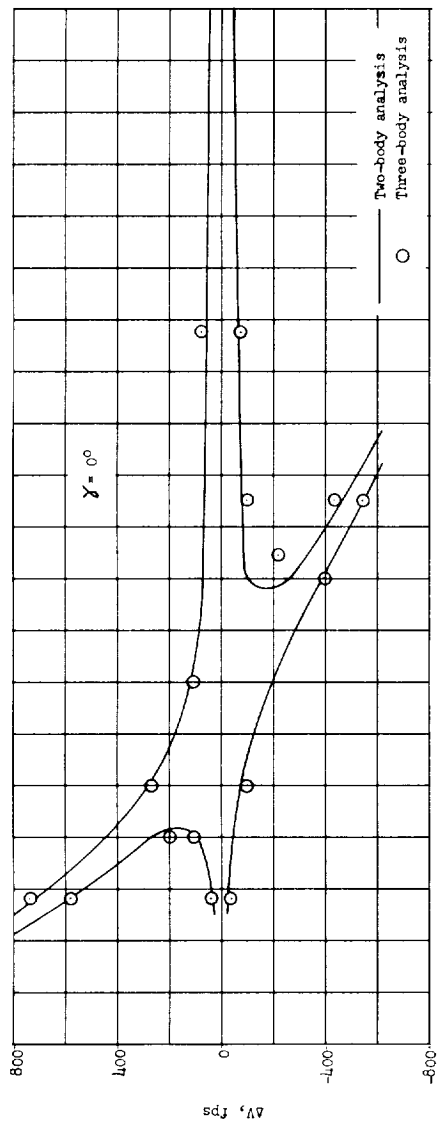


Figure 14.- Three-body and two-body impact error analyses. Two-dimensional case.
 $\gamma = 0^\circ$ and 25° .

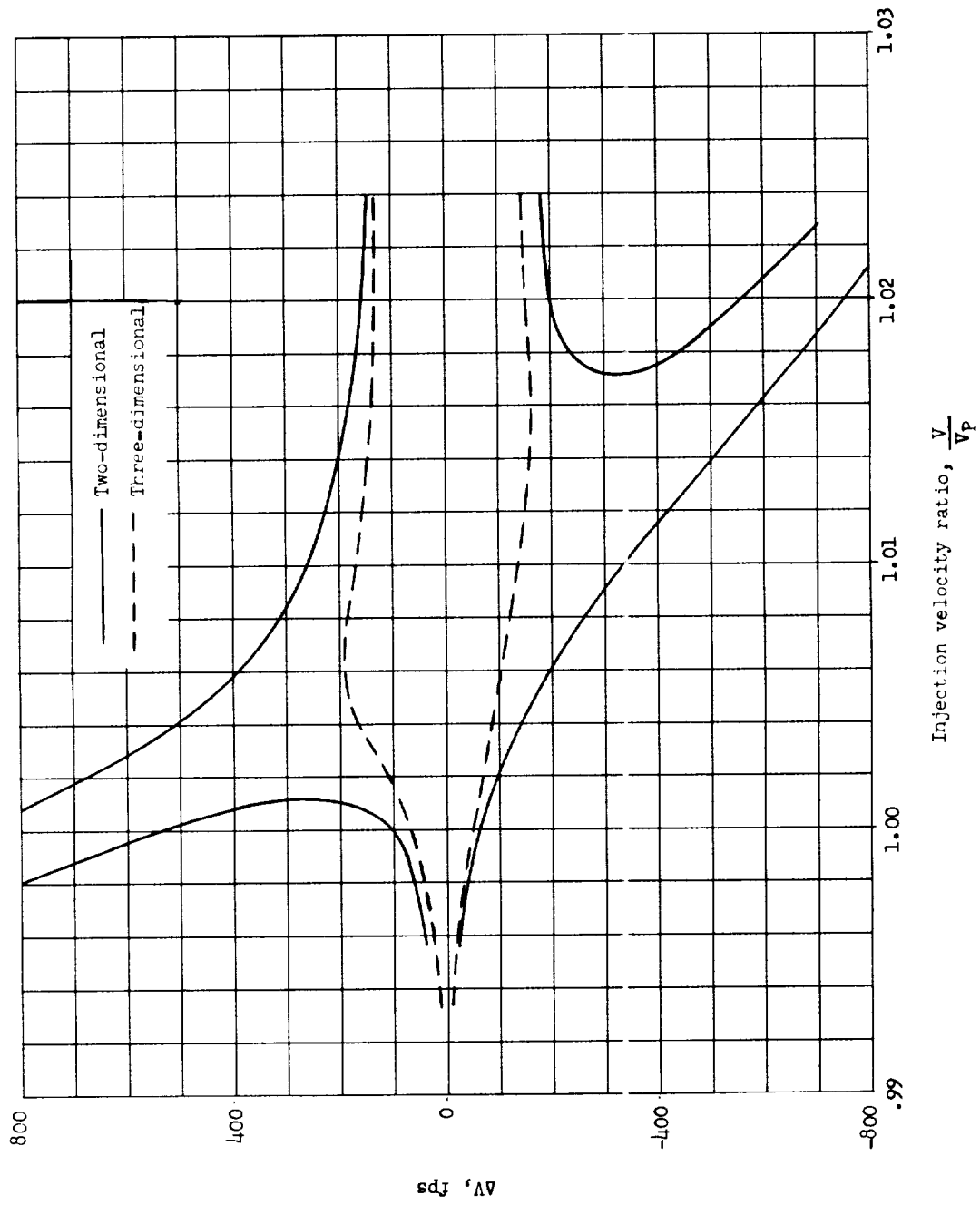


Figure 15.- Comparison of two- and three-dimensional impact error analyses. $\gamma = 25^\circ$.

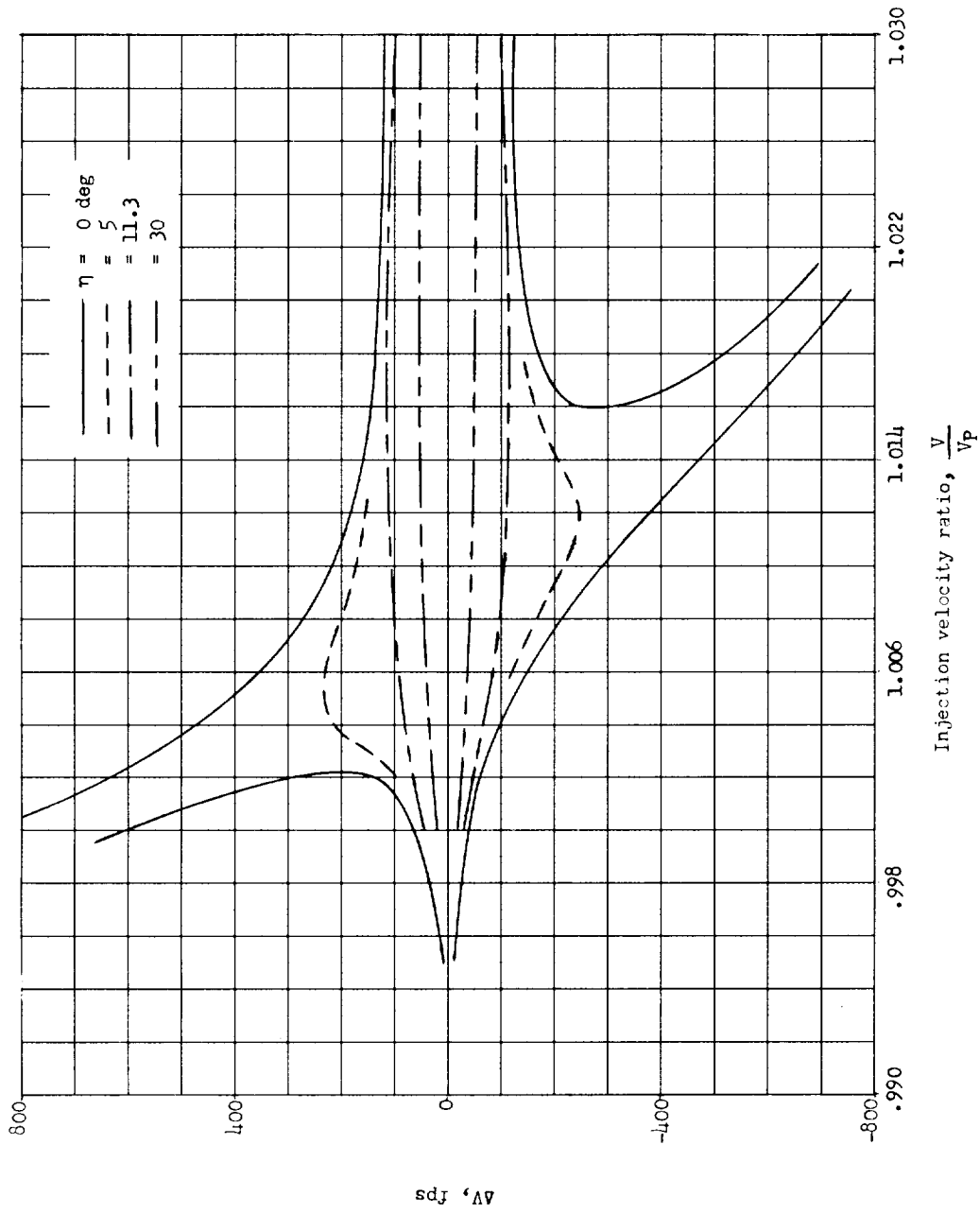


Figure 16.- Effect on velocity tolerance of inclination of plane of vehicle trajectory to earth-moon plane. $\gamma = 25^\circ$.

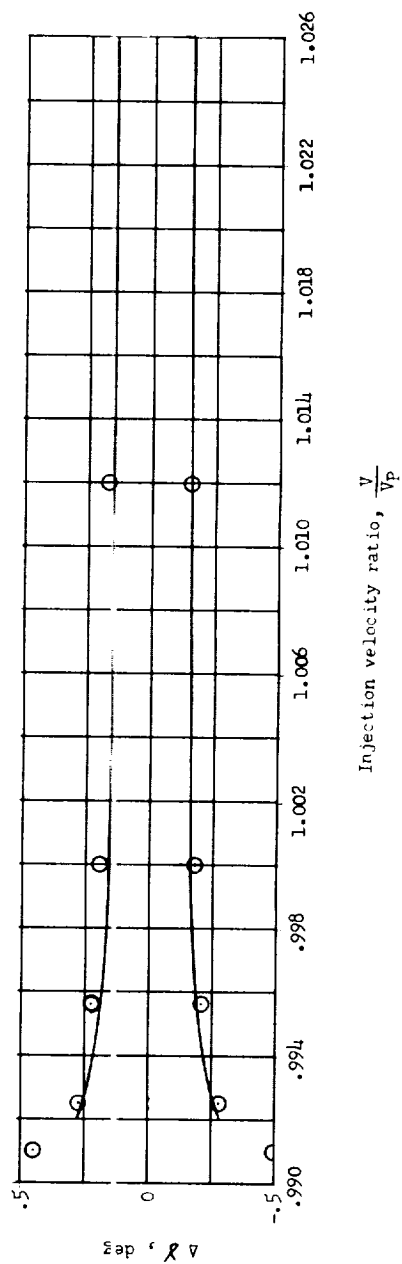
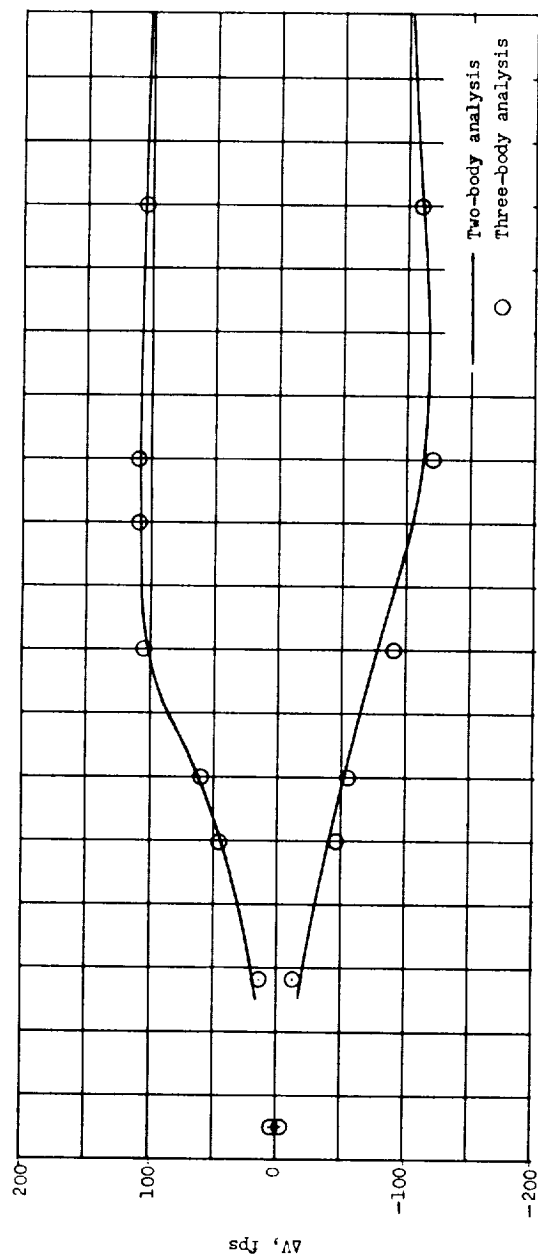


Figure 17.- Comparison of two- and three-body analyses for hitting within sphere below moon.
 $\gamma = 25^\circ$.